EXERCISE SOLUTIONS, LECTURES 15-20

Contents

15. Directional derivatives and the gradient

Exercise 1. Find the gradient of f.

(1) $f(x,y) = 3x^2y - xy^3$
(2) $f(x,y) = \frac{x}{x+y}$
(3) $f(x,y) = \sqrt{x^2+y^2}$
(4) $f(x,y) = x\ln(x) + y\ln(y)$
(5) $f(x,y) = e^{x\sin(y)}$
(6) $f(x,y,z) = \frac{x}{y+z}$
(7) $f(x,y,z) = x\ln(yz)$
(8) $f(x,y,z) = xyze^{xyz}$

Solution. (1)

$$\nabla f(x,y) = \langle 6xy - y^3, 3x^2 - 3xy^2\rangle$$

(2) We write this as $f(x,y) = x(x+y)^{-1}$. Then

$$f_x(x,y) = (x+y)^{-1} + x\frac{\partial}{\partial x}\left((x+y)^{-1}\right) = \frac{1}{x+y} - x(x+y)^{-2} = \frac{1}{x+y} - \frac{x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$f_y(x,y) = -x(x+y)^{-2} = -\frac{x}{(x+y)^2}$$

So

$$\nabla f(x,y) = \langle \frac{y}{(x+y)^2}, -\frac{x}{(x+y)^2}\rangle$$

(3) We write this as $f(x,y) = (x^2+y^2)^{1/2}$. Then

$$f_x(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2y) = \frac{y}{\sqrt{x^2+y^2}}$$

So

$$\nabla f(x,y) = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle$$

(4) We have

$$f_x(x,y) = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$
$$f_y(x,y) = \ln(y) + y \cdot \frac{1}{y} = \ln(y) + 1$$

So

$$\nabla f(x,y) = \langle \ln(x)+1, \ln(y)+1 \rangle$$

(5) We have by Chain Rule

$$f_x(x,y) = e^{x\sin(y)}\sin(y), \quad f_y(x,y) = e^{x\sin(y)}x\cos(y),$$

so

$$\nabla f(x,y) = \langle e^{x\sin(y)}\sin(y), e^{x\sin(y)}x\cos(y) \rangle$$

(6) We write this as $f(x,y,z) = x(y+z)^{-1}$. We have

$$f_x(x,y,z) = (y+z)^{-1} = \frac{1}{y+z}$$
$$f_y(x,y,z) = -x(y+z)^{-2} = -\frac{x}{(y+z)^2}$$
$$f_z(x,y,z) = -x(y+z)^{-2} = -\frac{x}{(y+z)^2}$$

So

$$\nabla f(x,y,z) = \langle \frac{1}{y+z}, -\frac{x}{(y+z)^2}, -\frac{x}{(y+z)^2} \rangle$$

(7) We have

$$f_x(x,y,z) = \ln(yz), \quad f_y(x,y,z) = x\frac{1}{yz} \cdot z = \frac{x}{y}, \quad f_z(x,y,z) = x\frac{yz}{\cdot}y = \frac{x}{z}$$

(8) We have

$$f_x(x,y,z) = yze^{xyz} + xyze^{xyz} \cdot (yz) = (yz + xy^2z^2)e^{xyz}$$
$$f_y(x,y,z) = xze^{xyz} + xyze^{xyz} \cdot (xz) = (xz + x^2yz^2)e^{xyz}$$
$$f_z(x,y,z) = xye^{xyz} + xyze^{xyz} \cdot (xy) = (xy + x^2y^2z)e^{xyz}$$

So

$$\nabla f(x,y,z) = \langle (yz + xy^2z^2)e^{xyz}, (xz + x^2yz^2)e^{xyz}, (xy + x^2y^2z)e^{xyz} \rangle$$

$\square$

Exercise 2. Find the directional derivative.

(1) $D_{\vec{u}}f(1,1)$, where $f(x,y) = x^2 + y^2$ and $\vec{u} = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$
(2) $D_{\vec{u}}f(3,0)$, where $f(x,y) = x^2e^y$ and $\vec{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$.

Solution. (1) Note $D_{\vec{u}}f(1,1) = \vec{u} \cdot \nabla f(1,1)$. We have

$$f_x(x,y) = 2x, \quad f_y(x,y) = 2y,$$

so

$$\nabla f(1,1) = \langle 2, 2 \rangle$$

so

$$D_{\vec{u}}f(1,1) = \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \cdot \langle 2, 2 \rangle = \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} = 0$$

(2) Note that $D_{\vec{u}}f(3,0) = \vec{u} \cdot \nabla f(3,0)$. We have

$$f_x(x,y) = 2xe^y, \quad f_y(x,y) = x^2e^y,$$

so

$$\nabla f(3,0) = \langle 6, 9 \rangle$$

so

$$D_{\vec{u}}f(3,0) = \langle \frac{3}{5}, -\frac{4}{5} \rangle \cdot \langle 6, 9 \rangle = \frac{18}{5} - \frac{36}{5} = -\frac{18}{5}$$

□

Exercise 3. Find the maximum rate of increase of f at the given point, and the direction in which it occurs.

(1) $f(x,y) = \sin(xy)$ at $(1,0)$.
(2) $f(x,y) = 2xy^2 + xy^3$ at $(1,2)$.
(3) $f(x,y,z) = xyz^2 + x^2y^2$ at $(1,0,-1)$

Solution. (1) The direction of maximum rate of increase is the unit vector in the direction of gradient, $\nabla f(1,0)$. Note

$$f_x(x,y) = y\cos(xy), \quad f_y(x,y) = x\cos(xy),$$

so

$$\nabla f(1,0) = \langle 0, 1 \rangle$$

Since this is already a unit vector, the direction of maximum rate of increase is $\langle 0, 1 \rangle$. The maximum rate of increase is $|\nabla f(1,0)| = 1$.

(2) The direction of maximum rate of increase is the unit vector in the direction of gradient, $\nabla f(1,2)$. Note

$$f_x(x,y) = 2y^2 + y^3, \quad f_y(x,y) = 4xy + 3xy^2$$

so

$$\nabla f(1,2) = \langle 2 \cdot 2^2 + 2^3, 4 \cdot 2 + 3 \cdot 2^2 \rangle = \langle 16, 20 \rangle$$

Thus the direction of maximum rate of increase is

$$\frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle 16, 20 \rangle}{\sqrt{16^2 + 20^2}} = \frac{\langle 16, 20 \rangle}{\sqrt{656}} = \frac{\langle 4, 5 \rangle}{\sqrt{41}} = \langle \frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}} \rangle$$

The maximum rate of increase is $|\nabla f(1,2)| = 4\sqrt{41}$.

(3) The direction of maximum rate of increase is the unit vector in the direction of gradient, $\nabla f(1,0,-1)$. Note

$$f_x(x,y,z) = yz^2 + 2xy^2, \quad f_y(x,y,z) = xz^2 + 2x^2y, \quad f_z(x,y,z) = 2xyz$$

so

$$\nabla f(1,0,-1) = \langle 0,1,0\rangle$$

This is already a unit vector, so $\langle 0,1,0\rangle$ is the direction of the maximum rate of increase. The maximum rate of increase is $|\nabla f(1,0,-1)| = 1$.

□

Exercise 4. Find the tangent plane.

(1) Tangent plane to $xyz = 6$ at $(1,2,3)$
(2) Tangent plane to $x+y+z = e^{xyz}$ at $(0,0,1)$
(3) Tangent plane to $x^4+y^4+z^4 = 3x^2y^2z^2$ at $(1,1,1)$

Solution. (1) The surface is the level surface $f(x,y,z) = 6$ where $f(x,y,z) = xyz$. The equation of the tangent plane is

$$f_x(1,2,3)(x-1) + f_y(1,2,3)(y-2) + f_z(1,2,3)(z-3) = 0$$

Note that

$$f_x(x,y,z) = yz, \quad f_y(x,y,z) = xz, \quad f_z(x,y,z) = xy$$

so

$$f_x(1,2,3) = 6, \quad f_y(1,2,3) = 3, \quad f_z(1,2,3) = 2$$

so the tangent plane has equation

$$6(x-1) + 3(y-2) + 2(z-3) = 0,$$

or

$$6x + 3y + 2z = 18$$

(2) The surface is the level surface $f(x,y,z) = 0$ where $f(x,y,z) = x+y+z-e^{xyz}$. The equation of the tangent plane is

$$f_x(0,0,1)(x-0) + f_y(0,0,1)(y-0) + f_z(0,0,1)(z-1) = 0$$

Note that

$$f_x(x,y,z) = 1 - yze^{xyz}, \quad f_y(x,y,z) = 1 - xze^{xyz}, \quad f_z(x,y,z) = 1 - xye^{xyz}$$

so

$$f_x(0,0,1) = 1, \quad f_y(0,0,1) = 1, \quad f_z(0,0,1) = 1$$

so the tangent plane has equation

$$x + y + z - 1 = 0$$

(3) The surface is the level surface $f(x,y,z) = 0$ where $f(x,y,z) = x^4+y^4+z^4-3x^2y^2z^2$. The equation of the tangent plane is

$$f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1) = 0$$

Note that

$$f_x(x,y,z) = 4x^3 - 6xy^2z^2, \quad f_y(x,y,z) = 4y^3 - 6x^2yz^2, \quad f_z(x,y,z) = 4z^3 - 6x^2y^2z$$

so

$$f_x(1,1,1) = -2, \quad f_y(1,1,1) = -2, \quad f_z(1,1,1) = -2$$

so the tangent plane has equation

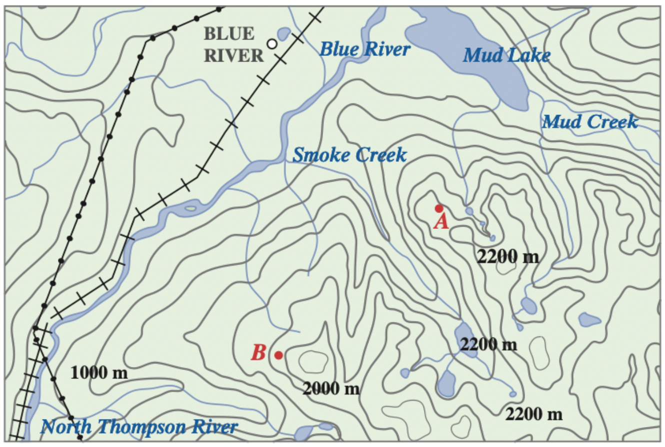
$$-2(x-1) - 2(y-1) - 2(z-1) = 0,$$

or

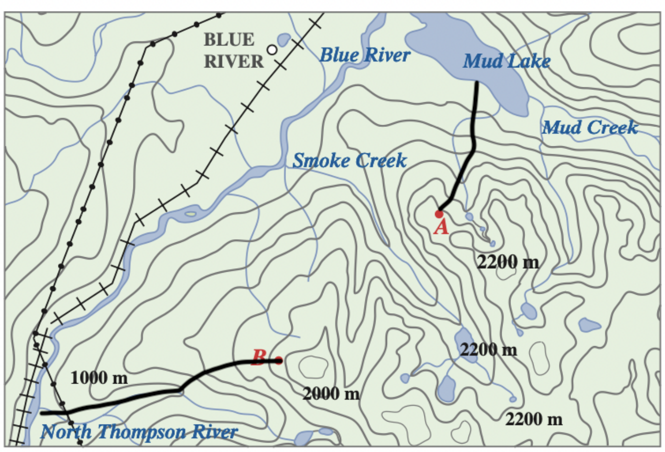
$$-2x - 2y - 2z = -6$$

$\square$

Exercise 5. Shown is a topographic map of Blue River Pine Provincial Park in British Columbia. Draw curves of steepest descent from point A (descending to Mud Lake) and from point B.



Solution. You follow the negative gradient vectors. The picture is just an approximation, so there might be certain inaccurancies in the drawing.



$\square$

16. Local maxima and minima, critical points

Exercise 1. Find the critical points and use the Second Derivative Test to determine whether they are local minima, local maxima or saddle points.

(1) $f(x,y) = xy - 2x - 2y - x^2 - y^2$
(2) $f(x,y) = y(e^x - 1)$
(3) $f(x,y) = 2 - x^4 + 2x^2 - y^2$
(4) $f(x,y) = (6x - x^2)(4y - y^2)$
(5) $f(x,y) = (x^2 + y^2)e^{-x}$
(6) $f(x,y) = \sin x \sin y$, in $-\pi < x < \pi$ and $-\pi < y < \pi$
(7) $f(x,y) = y^2 - 2y\cos x$, in $-1 \leq x \leq 7$ and $-3 \leq y \leq 3$
(8) $f(x,y) = -(x^2 - 1)^2 - (x^2 y - x - 1)^2$
(9) $f(x,y) = 3xe^y - x^3 - e^{3y}$

Solution. (1) We first find the critical points. Note

$$f_x(x,y) = y - 2 - 2x, \quad f_y(x,y) = x - 2 - 2y$$

So if (x,y) is a critical point, this means

$$y - 2 - 2x = 0, \quad x - 2 - 2y,$$

or

$$y = 2 + 2x, \quad x = 2 + 2y.$$

Plugging $x = 2 + 2y$ into $y = 2 + 2x$, we get

$$y = 2 + 2(2 + 2y) = 6 + 4y,$$

or $6 = -3y$, or $y = -2$. From this, we get $x = 2 - 4 = -2$. So there is only one critical point, $(-2,-2)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = -2, \quad f_{xy}(x,y) = 1, \quad f_{yy}(x,y) = -2,$$

so

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 = 4 - 1 = 3.$$

This is always positive. Note also that f_{xx} is always $-2 < 0$, so any critical point has to be local maximum.

(2) We first find the critical points. Note

$$f_x(x,y) = ye^x, \quad f_y(x,y) = e^x - 1$$

so if (x,y) is a critical point, this means

$$ye^x = 0, \quad e^x - 1 = 0.$$

Since $ye^x = 0$ means $y = 0$ or $e^x = 0$, and since e^x is never zero, this means $y = 0$. The second equation means $e^x = 1$, or $x = \ln(1) = 0$. Thus there is one critical point, $(0,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = ye^x, \quad f_{xy}(x,y) = e^x, \quad f_{yy}(x,y) = 0,$$

so

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 = -e^{2x}.$$

This is always negative, so any critical point is a saddle point.

(3) We first find the critical points. Note that

$$f_x(x,y) = -4x^3 + 4x, \quad f_y(x,y) = -2y.$$

So if (x,y) is a critical point, this means

$$-4x^3 + 4x = 0, \quad -2y = 0.$$

So first of all $y = 0$, and we have $-4x(x-1)(x+1) = 0$. Thus x could be either $0, -1$ or 1. The critical points are $(0,0)$, $(-1,0)$ and $(1,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = -12x^2 + 4, \quad f_{xy}(x,y) = 0, \quad f_{yy}(x,y) = -2,$$

so

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 = 24x^2 - 8$$

Thus

$$D(0,0) = -8 < 0$$

which means that $(0,0)$ is a saddle point. Also,

$$D(-1,0) = 24 - 8 > 0, \quad f_{xx}(-1,0) = -12 + 4 < 0,$$

which means that $(-1,0)$ is a local maximum.

$$D(1,0) = 24 - 8 > 0, \quad f_{xx}(1,0) = -12 + 4 < 0,$$

which means that $(1,0)$ is a local maximum.

(4) We first find the critical points. Note that

$$f_x(x,y) = (6-2x)(4y - y^2), \quad f_y(x,y) = (6x - x^2)(4 - 2y),$$

so if (x,y) is a critical point, it means

$$(6-2x)(4y-y^2) = 0, \quad (6x - x^2)(4-2y) = 0.$$

The first equation means that either $6 - 2x = 0$ or $4y - y^2 = 0$. Note also that $6 - 2x = 0$ means $x = 3$, and $4y - y^2 = 0$ means either $y = 0$ or $y = 4$. So the first requirement is either $x = 3$, $y = 0$ or $y = 4$.

The second equation means that either $6x - x^2 = 0$ or $4 - 2y = 0$. Note also that $6x - x^2 = 0$ means either $x = 0$ or $x = 6$, and $4 - 2y = 0$ means $y = 2$. So the second requirement is either $x = 0$, $x = 6$ or $y = 2$.

So a pair (x,y) satisfying the two requirements are as follows, following the first requirement first:

- If $x = 3$, then out of the three possible outcomes of the second requirement, $x = 0$, $x = 6$ or $y = 2$, the only possibility is $y = 2$, so $(3,2)$.
- If $y = 0$, then out of the three possible outcomes of the second requirement, $x = 0$, $x = 6$ or $y = 2$, it could possibly be either $x = 0$ or $x = 6$, so $(0,0)$ or $(6,0)$.
- If $y = 4$, then out of the three possible outcomes of the second requirement, $x = 0$, $x = 6$ or $y = 2$, it could possibly be either $x = 0$ or $x = 6$, so $(0,4)$ or $(6,4)$.

So the critical points are $(3,2)$, $(0,0)$, $(6,0)$, $(0,4)$ and $(6,4)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = -2(4y - y^2), \quad f_{xy}(x,y) = (6-2x)(4-2y), \quad f_{yy}(x,y) = -2(6x - x^2)$$

so

$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y) = 4(4y-y^2)(6x-x^2) - (6-2x)^2(4-2y)^2$$

We apply the Second Derivative Test to the five critical points.

- If $(x,y) = (3,2)$, then
$$D(3,2) = 4(8-4)(18-9) - 0 > 0, \quad f_{xx}(x,y) = -2(8-4) < 0,$$
so $(3,2)$ is a local maximum.
- If $(x,y) = (0,0)$, then
$$D(0,0) = 0 - 6^2 4^2 < 0,$$
so $(0,0)$ is a saddle point.
- If $(x,y) = (6,0)$, then
$$D(6,0) = 0 - (6-12)^2 4^2 < 0,$$
so $(6,0)$ is a saddle point.
- If $(x,y) = (0,4)$, then
$$D(0,4) = 0 - 6^2(4-8)^2 < 0,$$
so $(0,4)$ is a saddle point.
- If $(x,y) = (6,4)$, then
$$D(6,4) = 0 - (6-12)^2(4-8)^2 < 0,$$
so $(6,4)$ is a saddle point.

(5) We first find the critical points. Note that
$$f_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x} = (2x - x^2 - y^2)e^{-x}, \quad f_y(x,y) = 2ye^{-x},$$
so if (x,y) is a critical point, it means that
$$(2x - x^2 - y^2)e^{-x} = 0, \quad 2ye^{-x} = 0.$$
Since e^{-x} is never zero, this means
$$2x - x^2 - y^2 = 0, \quad 2y = 0.$$
So $y = 0$, and $2x - x^2 = 0$, which means either $x = 0$ or $x = 2$. So the critcial points are $(0,0)$ and $(2,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that
$$f_{xx}(x,y) = (2-2x)e^{-x} - (2x - x^2 - y^2)e^{-x} = (2 - 4x + x^2 + y^2)e^{-x},$$
$$f_{xy}(x,y) = -2ye^{-x},$$
$$f_{yy}(x,y) = 2e^{-x}.$$
So
$$D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 = (4 - 8x + 2x^2 + 2y^2)e^{-2x} - 4y^2e^{-2x}$$

- For the critical point $(0,0)$, we have
$$D(0,0) = 4 > 0, \quad f_{xx}(0,0) = 2 > 0,$$
so $(0,0)$ is a local minimum.

- For the critical point $(2,0)$, we have
$$D(2,0) = (4-16+8)e^{-4} < 0,$$
so $(2,0)$ is a saddle point.

(6) We first find the critical points. Note that
$$f_x(x,y) = \cos x \sin y, \quad f_y(x,y) = \sin x \cos y,$$
so if (x,y) is a critical point, it means
$$\cos x \sin y = 0, \quad \sin x \cos y = 0.$$
So the first requirement is either $\cos x = 0$ or $\sin y = 0$, and the second requirement is either $\sin x = 0$ or $\cos y = 0$.
- If $\cos x = 0$, then $\sin x \neq 0$, so $\cos y = 0$.
- If $\sin y = 0$, then $\cos y \neq 0$, so $\sin x = 0$.

So (x,y) is a critical point if either $\cos x = \cos y = 0$ or $\sin x = \sin y = 0$.
- If $\cos x = \cos y = 0$, then it means x, y are either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. So the critical points coming out of this possibility are $(-\frac{\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{\pi}{2})$.
- If $\sin x = \sin y = 0$, then it means x, y are both 0, so the critical point coming out of this possibility is $(0,0)$.

So the critcial points in the region are $(-\frac{\pi}{2}, -\frac{\pi}{2})$, $(-\frac{\pi}{2}, \frac{\pi}{2})$, $(\frac{\pi}{2}, -\frac{\pi}{2})$, $(\frac{\pi}{2}, \frac{\pi}{2})$ and $(0,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that
$$f_{xx}(x,y) = -\sin x \sin y, \quad f_{xy}(x,y) = \cos x \cos y, \quad f_{yy}(x,y) = -\sin x \sin y,$$
so
$$D(x,y) = \sin^2 x \sin^2 y - \cos^2 x \cos^2 y.$$
- For the critical point $(-\frac{\pi}{2}, -\frac{\pi}{2})$, we have
$$D(-\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0, \quad f_{xx}(-\frac{\pi}{2}, -\frac{\pi}{2}) = -1 < 0,$$
so $(-\frac{\pi}{2}, -\frac{\pi}{2})$ is a local maximum.
- For the critical point $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have
$$D(-\frac{\pi}{2}, \frac{\pi}{2}) = 1 > 0, \quad f_{xx}(-\frac{\pi}{2}, \frac{\pi}{2}) = 1 > 0,$$
so $(-\frac{\pi}{2}, \frac{\pi}{2})$ is a local minimum,
- For the critical point $(\frac{\pi}{2}, -\frac{\pi}{2})$, we have
$$D(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0, \quad f_{xx}(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0,$$
so $(\frac{\pi}{2}, -\frac{\pi}{2})$ is a local minimum.
- For the critical point $(\frac{\pi}{2}, -\frac{\pi}{2})$, we have
$$D(\frac{\pi}{2}, -\frac{\pi}{2}) = 1 > 0, \quad f_{xx}(\frac{\pi}{2}, -\frac{\pi}{2}) = -1 < 0,$$
so $(\frac{\pi}{2}, -\frac{\pi}{2})$ is a local maximum.
- For the critical point $(0,0)$, we have
$$D(0,0) = -1 < 0,$$
so $(0,0)$ is a saddle point.

(7) We first find the critical points. Note that

$$f_x(x,y) = 2y\sin x, \quad f_y(x,y) = 2y - 2\cos x,$$

so if (x,y) is a critical point, we have

$$2y\sin x = 0, \quad 2y - 2\cos x = 0.$$

So $y = \cos x$, and either $y = 0$ or $\sin x = 0$. If $y = 0$, then $\cos x = 0$, which means that $x = \cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$. Since $-\frac{\pi}{2} \sim -1.57$, $\frac{3\pi}{2} \sim 4.71$, $\frac{5\pi}{2} \sim 7.85$, the points in the range $-1 \leq x \leq 7$ are $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. If $\sin x = 0$, then $\cos x$ could be either 1 or -1, so $y = 1$ or -1. Note also that $\sin x = 0$ in the range $-1 \leq x \leq 7$ means $x = 0, \pi$ or 2π, because $3\pi \sim 9.42 > 7$ and $-\pi \sim -3.14 < -1$. So the critical points are $(\frac{\pi}{2}, 0)$, $(\frac{3\pi}{2}, 0)$, $(0,1)$, $(0,-1)$, $(\pi,1)$, $(\pi,-1)$, $(2\pi,1)$, $(2\pi,-1)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that

$$f_{xx}(x,y) = 2y\cos x, \quad f_{xy}(x,y) = 2\sin x, \quad f_{yy}(x,y) = 2$$

So

$$D(x,y) = 4y\cos x - 4\sin^2 x.$$

- For the critical point $(\frac{\pi}{2}, 0)$, we have
$$D(\frac{\pi}{2}, 0) = -4 < 0,$$
so $(\frac{\pi}{2}, 0)$ is a saddle point.
- For the critical point $(\frac{3\pi}{2}, 0)$,
$$D(\frac{3\pi}{2}, 0) = -4 < 0,$$
so $(\frac{3\pi}{2}, 0)$ is a saddle point.
- For the critical point $(0,1)$, we have
$$D(0,1) = 4 > 0, \quad f_{xx}(0,1) = 2 > 0,$$
so $(0,1)$ is a local minimum.
- For the critical point $(0,-1)$, we have
$$D(0,-1) = -4 < 0,$$
so $(0,-1)$ is a saddle point.
- For the critical point $(\pi,1)$,
$$D(\pi,1) = -4 < 0,$$
so $(\pi,1)$ is a saddle point.
- For the critical point $(\pi,-1)$,
$$D(\pi,-1) = 4 > 0, \quad f_{xx}(\pi,-1) = 2 > 0,$$
so $(\pi,-1)$ is a local minimum.
- For the critical point $(2\pi,1)$,
$$D(2\pi,1) = 4 > 0, \quad f_{xx}(2\pi,1) = 2 > 0,$$
so $(2\pi,1)$ is a local minimum.

- For the critical point $(2\pi, -1)$,
$$D(2\pi, -1) = -4 < 0,$$
so $(2\pi, -1)$ is a saddle point.

(8) We first find the critical points. Note that
$$f_x(x, y) = -2(x^2-1)\cdot(2x)-2(x^2y-x-1)\cdot(2xy-1) = -4x(x^2-1)-2(2xy-1)(x^2y-x-1),$$
$$f_y(x, y) = -2(x^2y - x - 1) \cdot (x^2) = -2x^2(x^2y - x - 1).$$
So if (x, y) is a critical point, this means
$$-4x(x^2 - 1) - 2(2xy - 1)(x^2y - x - 1) = 0, \quad -2x^2(x^2y - x - 1) = 0.$$
The second requirement means either $x = 0$ or $x^2y - x - 1 = 0$.

- If $x = 0$, then the first requirement becomes
$$-2(-1)(-1) = 0,$$
which is absurd.
- If $x^2y - x - 1 = 0$, then the first requirement becomes
$$-4x(x^2 - 1) = 0,$$
so either $x = 0$, $x = 1$ or $x = -1$.
  - If $x = 0$, then $x^2y - x - 1 = 0$ becomes $-1 = 0$, which is absurd.
  - If $x = 1$, then $x^2y - x - 1 = 0$ becomes $y - 2 = 0$, or $y = 2$. So $(1, 2)$ is a critical point.
  - If $x = -1$, then $x^2y - x - 1 = 0$ becomes $y = 0$, so $(-1, 0)$ is a critical point.

So the critical points are $(1, 2)$ and $(-1, 0)$.

To use the Second Derivative Test, we need to compute what $D(x, y)$ is. Note that
$$\begin{aligned} f_{xx}(x, y) &= -4(x^2 - 1) - 4x \cdot (2x) - 2(2y)(x^2y - x - 1) - 2(2xy - 1)(2xy - 1) \\ &= -4(x^2 - 1) - 8x^2 - 4y(x^2y - x - 1) - 2(2xy - 1)^2 \\ f_{xy}(x, y) &= -2(2x)(x^2y - x - 1) - 2(2xy - 1)x^2 \\ f_{yy}(x, y) &= -2x^4. \end{aligned}$$
Note that for both $(x, y) = (1, 2)$ and $(-1, 0)$, we had $x^2y - x - 1 = 0$ and $x^2 = 1$. Using this, we have
$$f_{xx}(1, 2) = -8 - 2(4 - 1)^2 = -8 - 18 = -26,$$
$$f_{xy}(1, 2) = -2(4 - 1) = -6,$$
$$f_{yy}(1, 2) = -2,$$
so $D(1, 2) = 52 - 36 > 0$, and $f_{xx}(1, 2) < 0$, so $(1, 2)$ is a local maximum. For $(-1, 0)$, we have
$$f_{xx}(-1, 0) = -8 - 2(-1)^2 = -10,$$
$$f_{xy}(-1, 0) = -2(-1) = 2,$$
$$f_{yy}(-1, 0) = -2,$$
so $D(-1, 0) = 20 - 4 > 0$, and $f_{xx}(-1, 0) < 0$, so $(-1, 0)$ is a local maximum.

(9) We first find the critical points. Note that
$$f_x(x,y) = 3e^y - 3x^2, \quad f_y(x,y) = 3xe^y - 3e^{3y},$$
so if (x,y) is a critical point, it means
$$3e^y - 3x^2 = 0, \quad 3xe^y - 3e^{3y} = 0.$$
The second requirement says $3xe^y = 3e^{3y}$, or $x = e^{2y}$. The first requirement says $3e^y = 3x^2$, or $e^y = x^2$. Thus,
$$x = e^{2y} = (e^y)^2 = (x^2)^2 = x^4.$$
This means either $x = 0$ or $x^3 = 1$, or $x = 1$. If $x = 0$, then $e^{2y} = 0$, which is absurd. If $x = 1$, then $e^{2y} = 1$, so $y = 0$. Thus there is only one critical point, $(1,0)$.

To use the Second Derivative Test, we need to compute what $D(x,y)$ is. Note that
$$f_{xx}(x,y) = -6x, \quad f_{xy}(x,y) = 3e^y, \quad f_{yy}(x,y) = 3xe^y - 9e^{3y}.$$
So
$$f_{xx}(1,0) = -6, \quad f_{xy}(1,0) = 3, \quad f_{yy}(1,0) = -6.$$
So
$$D(1,0) = 36 - 9 > 0, \quad f_{xx}(1,0) = -6 < 0,$$
so $(1,0)$ is a local maximum.

□

17. Global maxima and minima

Exercise 1. Find the global maximum minimum values of $f(x,y)$ on the region D.

(1) $f(x,y) = x^2 + y^2 - 2x$, and D is the triangular region with vertices $(2,0)$, $(0,2)$ and $(0,-2)$, including boundaries.
(2) $f(x,y) = x + y - xy$, and D is the triangular region with vertices $(0,0)$, $(0,2)$, and $(4,0)$, including boundaries.
(3) $f(x,y) = x^2 + y^2 + x^2y + 4$, and $D = \{(x,y) \mid |x| \leq 1,\ |y| \leq 1\}$.
(4) $f(x,y) = x^2 + xy + y^2 - 6y$, and $D = \{(x,y) \mid -3 \leq x \leq 3,\ 0 \leq y \leq 5\}$.
(5) $f(x,y) = x^2 + 2y^2 - 2x - 4y + 1$, and $D = \{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 3\}$.

Solution. (1) We first find the critical points. Note that
$$f_x(x,y) = 2x - 2, \quad f_y(x,y) = 2y,$$
so if (x,y) is a critical point, then $2x - 2 = 0$ and $2y = 0$, which means that $x = 1$ and $y = 0$. Since $(1,0)$ is inside the region D, there is a critical point and it is $(1,0)$. The value of $f(x,y)$ at the critical point $(1,0)$ is $f(1,0) = -1$.

The boundary of the region D is naturally divided into three parts.

- $L_1 = \{(x, 2-x) \mid 0 \leq x \leq 2\}$, the line connecting $(2,0)$ and $(0,2)$: on L_1, the function $f(x,y)$ becomes
$$f(x, 2-x) = x^2 + (2-x)^2 - 2x = 2x^2 - 6x + 4$$
Finding the maximum and the minimum values of f on L_1 amounts to finding the global maximum and minimum values of $f(x) = 2x^2 - 6x + 4$ on $0 \leq x \leq 2$. This function has the critical point when $f'(x) = 4x - 6 = 0$, or $x = \frac{3}{2}$. At this point, the value of f is $f(\frac{3}{2}) = 2 \cdot \frac{9}{4} - 6 \cdot \frac{3}{2} + 4 = -\frac{1}{2}$. There are two boundary points, $x = 0$

and $x = 2$, and at them the values of f are $f(0) = 4$ and $f(2) = 8 - 12 + 4 = 0$. Thus, among those values, the largest is 4 and the smallest is $-\frac{1}{2}$. Thus, on L_1, the maximum value of f is 4, and the minimum value of f is $-\frac{1}{2}$.

- $L_2 = \{(0, y) \mid -2 \leq y \leq 2\}$, the line connecting $(0, 2)$ and $(0, -2)$: on L_2, the function $f(x, y)$ becomes
$$f(0, y) = y^2$$
Finding the maximum and the minimum values of f on L_2 amounts to finding the global maximum and minimum values of $f(y) = y^2$ on $-2 \leq y \leq 2$. This function has the critical point when $f'(y) = 2y = 0$, or $y = 0$. At this point, the value of f is $f(0) = 0$. There are two boundary pointst, $y = -2$ and $y = 2$, and at them the values of f are $f(-2) = f(2) = 4$. Thus, among those values, the largest is 4 and the smallest is 0. Thus, on L_2, the maximum value of f is 4, and the minimum value of f is 0.
- $L_3 = \{(x, x - 2) \mid 0 \leq x \leq 2\}$, the line connecting $(0, -2)$ and $(2, 0)$: on $L3$, the function $f(x, y)$ becomes
$$f(x, x - 2) = x^2 + (x - 2)^2 - 2x = 2x^2 - 6x + 4$$
Finding the maximum and the minimum values of f on L_3 amounts to finding the global maximum and minimum values of $f(x) = 2x^2 - 6x + 4$ on $0 \leq x \leq 2$. This is exactly the same problem as the problem on L_1, so we know that, on L_3, the maximum value of f is 4 and the minimum value of f is $-\frac{1}{2}$.

Among all these values, the maximum value is 4, and the minimum value is -1.

(2) We first find the critical points. Note that
$$f_x(x, y) = 1 - y, \quad f_y(x, y) = 1 - x$$
so if (x, y) is a critical point, then $1 - y = 0$ and $1 - x = 0$, which means that $x = 1$ and $y = 1$. Since $(1, 1)$ is inside the region D, there is a critcial point and it is $(1, 1)$. The value of $f(x, y)$ at the critical point $(1, 1)$ is $f(1, 1) = -1$.

The boundary of the region D is naturally divided into three parts.

- $L_1 = \{(0, y) \mid 0 \leq y \leq 2\}$, the line connecting $(0, 0)$ and $(0, 2)$: on L_1, the function $f(x, y)$ becomes
$$f(0, y) = y$$
Finding the maximum and the minimum values of f on L_1 amounts to finding the global maximum and minimum values of $f(y) = y$ on $0 \leq y \leq 2$. Obviously, the minimum value is 0 and the maximum value is 2.
- $L_2 = \{(4 - 2y, y) \mid 0 \leq y \leq 2\}$, the line connecting $(0, 2)$ and $(4, 0)$: on L_2, the function $f(x, y)$ becomes
$$f(4 - 2y, y) = (4 - 2y) + y - (4 - 2y)y = 4 - 5y + 2y^2$$
Finding the maximum and the minimum values of f on L_2 amounts to finding the global maximum and minimum values of $f(y) = 4 - 5y + 2y^2$ on $0 \leq y \leq 2$. This function has the critical point when $f'(y) = 4y - 5 = 0$, or $y = \frac{5}{4}$. At this point, the value of f is $f(\frac{5}{4}) = 4 - 5 \cdot \frac{5}{4} + 2 \cdot \frac{25}{16} = \frac{7}{8}$. There are two boundary pointst, $y = 0$ and $y = 2$, and at them the values of f are $f(0) = 4$ and $f(2) = 4 - 10 + 8 = 2$. Thus,

among those values, the largest is 4 and the smallest is $\frac{7}{8}$. Thus, on L_2, the maximum value of f is 4, and the minimum value of f is $\frac{7}{8}$.

- $L_3 = \{(x,0) \mid 0 \le x \le 4\}$, the line connecting $(0,0)$ and $(4,0)$: on $L3$, the function $f(x,y)$ becomes
$$f(x,0) = x$$
Finding the maximum and the minimum values of f on L_3 amounts to finding the global maximum and minimum values of $f(x) = x$ on $0 \le x \le 4$. Obviously, the minimum value is 0, and the maximum value is 4.

Among all these values, the maximum value is 4, and the minimum value is -1.

(3) We first find the critical points. Note that
$$f_x(x,y) = 2x + 2xy = 2x(1+y), \quad f_y(x,y) = 2y + x^2$$
so if (x,y) is a critical point, then $2x(1+y) = 0$ and $2y + x^2 = 0$. The first requirement says either $x = 0$ or $1 + y = 0$, or $y = -1$.

- If $x = 0$, the second requirement says that $2y = 0$, or $y = 0$. So $(0,0)$ is a critical point.
- If $y = -1$, the second requirement says that $-2 + x^2 = 0$, or $x^2 = 2$. Thus either $x = \sqrt{2}$ or $-\sqrt{2}$. On the other hand, $|\sqrt{2}| > 1$, so these critcial points do not belong to the region D.

So, there is one critical point, $(0,0)$. The value of $f(x,y)$ at the critical point is $f(0,0) = 4$.

The boundary of the region D, which is a square, is naturally divided into four parts.

- $L_1 = \{(-1,y) \mid -1 \le y \le 1\}$: on L_1, the function $f(x,y)$ becomes
$$f(-1,y) = 1 + y^2 + y + 4 = y^2 + y + 5$$
Finding the maximum and the minimum values of f on L_1 amounts to finding the global maximum and minimum values of $f(y) = y^2 + y + 5$ on $-1 \le y \le 1$. The critical point of $f(y)$ happens when $f'(y) = 2y + 1 = 0$, or $y = -\frac{1}{2}$. At the critcial point, the value of f is $f(-\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} + 5 = \frac{19}{4}$. There are two boundary points, $y = -1$ and $y = 1$, and at them the values of f are $f(-1) = 5$ and $f(1) = 7$. So, on L_1, the maximum value of f is 7, and the minimum value of f is $\frac{19}{4}$.
- $L_2 = \{(x,1) \mid -1 \le x \le 1\}$: on L_2, the function $f(x,y)$ becomes
$$f(x,1) = x^2 + 1 + x^2 + 4 = 2x^2 + 5$$
Finding the maximum and the minimum values of f on L_2 amounts to finding the global maximum and minimum values of $f(x) = 2x^2 + 5$ on $-1 \le x \le 1$. This function has the critical point when $f'(x) = 4x = 0$, or $x = 0$. At this point, the value of f is $f(0) = 5$. There are two boundary points, $x = -1$ and $x = 1$, and at them the values of f are $f(-1) = f(1) = 7$. Thus, among those values, the largest is 7 and the smallest is 5. Thus, on L_2, the maximum value of f is 7, and the minimum value of f is 5.
- $L_3 = \{(1,y) \mid -1 \le y \le 1\}$: on $L3$, the function $f(x,y)$ becomes
$$f(1,y) = 1 + y^2 + y + 4 = y^2 + y + 5$$
Finding the maximum and the minimum values of f on L_3 amounts to finding the global maximum and minimum values of $f(y) = y^2 + y + 5$ on $-1 \le y \le 1$. This

is exactly the same situation as that over L_1, so we know that, on L_3, the maximum value of f is 7, and the minimum value of f is $\frac{19}{4}$.

- $L_4 = \{(x,-1) \mid -1 \le x \le 1\}$: on L_4, the function $f(x,y)$ becomes
$$f(x,-1) = x^2 + 1 - x^2 + 4 = 5$$
So the maximum and minimum of f on L_4 are both 5.

Among all these values, the maximum value is 7, and the minimum value is 4.

(4) We first find the critical points. Note that
$$f_x(x,y) = 2x + y, \quad f_y(x,y) = x + 2y - 6,$$
so if (x,y) is a critical point, then $2x+y=0$ and $x+2y-6=0$. To solve this system of linear equations, we make the first equation into $y=-2x$, and plug this into the second equation, which yields $x+2(-2x)-6=0$, or $-3x-6=0$, or $x=-2$. From this, $y=4$. Since $(-2,4)$ is in the region of D, $(-2,4)$ is a critical point. The value of $f(x,y)$ on this point is $f(-2,4) = 4-8+16-24 = -12$.

The boundary of D, which is a rectangle, is naturally divided into four parts.

- $L_1 = \{(-3,y) \mid 0 \le y \le 5\}$: on L_1, the function $f(x,y)$ becomes
$$f(-3,y) = 9 - 3y + y^2 - 6y = y^2 - 9y + 9$$
Finding the maximum and the minimum values of f on L_1 amounts to finding the global maximum and minimum values of $f(y) = y^2 - 9y + 9$ on $0 \le y \le 5$. This function has the critical point when $f'(y) = 2y - 9 = 0$, or $y = \frac{9}{2}$. At this point, the value of f is $f(\frac{9}{2}) = \frac{81}{4} - \frac{81}{2} + 9 = -\frac{45}{4}$. The boundary points are $y=0$ and $y=5$, at which the values of f are $f(0)=9$ and $f(5) = 25 - 45 + 9 = -11$. Thus, on L_1, the maximum value of f is 9, and the minimum value of f is $-\frac{45}{4}$.
- $L_2 = \{(x,5) \mid -3 \le x \le 3\}$: on L_2, the function $f(x,y)$ becomes
$$f(x,5) = x^2 + 5x + 25 - 30 = x^2 + 5x - 5$$
Finding the maximum and the minimum values of f on L_2 amounts to finding the global maximum and minimum values of $f(x) = x^2 + 5x - 5$ on $-3 \le x \le 3$. This function has the critical point when $f'(x) = 2x+5 = 0$, or $x = -\frac{5}{2}$. At this point, the value of f is $f(-\frac{5}{2}) = \frac{25}{4} - \frac{25}{2} - 5 = -\frac{45}{4}$. The boundary points are $x=-3$ and $x=3$, at which the values of f are $f(-3) = 9 - 15 - 5 = -11$ and $f(3) = 9 + 15 - 5 = 19$. Thus, on L_2, the maximum value of f is 19, and the minimum value of f is $-\frac{45}{4}$.
- $L_3 = \{(3,y) \mid 0 \le y \le 5\}$: on L_3, the function $f(x,y)$ becomes
$$f(3,y) = 9 + 3y + y^2 - 6y = y^2 - 3y + 9$$
Finding the maximum and the minimum values of f on L_3 amounts to finding the global maximum and minimum values of $f(y) = y^2 - 3y + 9$. This function has the critical point when $f'(y) = 2y - 3 = 0$, or $y = \frac{3}{2}$. At this point, the value of f is $f(\frac{3}{2}) = \frac{9}{4} - \frac{9}{2} + 9 = \frac{27}{4}$. The boundary points are $y=0$ and $y=5$, at which the values of f are $f(0) = 9$ and $f(5) = 25 - 15 + 9 = 19$. Thus, on L_3, the maximum value of f is 19, and the minimum value of f is $\frac{27}{4}$.
- $L_4 = \{(x,0) \mid -3 \le x \le 3\}$: on L_4, the function $f(x,y)$ becomes
$$f(x,0) = x^2$$

Finding the maximum and the minimum values of f on L_4 amounts to finding the global maximum and minimum values of $f(x) = x^2$ on $-3 \leq x \leq 3$. This function has the critical point when $f'(x) = 2x = 0$, or $x = 0$. At this point, the value of f is $f(0) = 0$. There are two boundary points, $x = -3$ and $x = 3$, at which the values of f are $f(-3) = f(3) = 9$. Thus, on L_4, the maximum value of f is 9, and the minimum value of f is 0.

Combining these, the maximum value of f is 19, and the minimum value of f is -12.

(5) We first find the critical points. Note that

$$f_x(x, y) = 2x - 2, \quad f_y(x, y) = 4y - 4$$

so if (x, y) is a critical point, then $2x - 2 = 0$ and $4y - 4 = 0$, or $x = 1$ and $y = 1$. This is in the region of D, so $(1, 1)$ is the critical point, at which the value of f is $f(1, 1) = 1 + 2 - 2 - 4 + 1 = -2$.

The boundary of D, which is a rectangle, is naturally divided into four parts.

- $L_1 = \{(0, y) \mid 0 \leq y \leq 3\}$: on L_1, the function $f(x, y)$ becomes

$$f(0, y) = 2y^2 - 4y + 1$$

Finding the maximum and the minimum values of f on L_1 amounts to finding the global maximum and minimum values of $f(y) = 2y^2 - 4y + 1$ on $0 \leq y \leq 3$. This function has the critical point when $f'(y) = 4y - 4 = 0$, or $y = 1$. At this point, the value of f is $f(1) = 2 - 4 + 1 = -1$. There are two boundary points, $y = 0$ and $y = 3$, at which the values of f are $f(0) = 1$ and $f(3) = 18 - 12 + 1 = 7$. Thus, on L_1, the maximum value of f is 7, and the minimum value of f is -1.

- $L_2 = \{(x, 3) \mid 0 \leq x \leq 2\}$: on L_2, the function $f(x, y)$ becomes

$$f(x, 3) = x^2 + 18 - 2x - 12 + 1 = x^2 - 2x + 7$$

Finding the maximum and the minimum values of f on L_2 amounts to finding the global maximum and minimum values of $f(x) = x^2 - 2x + 7$ on $0 \leq x \leq 2$. This function has the critical point when $f'(x) = 2x - 2 = 0$, or $x = 1$. At this point, the value of f is $f(1) = 1 - 2 + 7 = 6$. There are two boundary points, $x = 0$ and $x = 2$, at which the values of f are $f(0) = 7$ and $f(2) = 4 - 4 + 7 = 7$. Thus, on L_2, the maximum value of f is 7, and the minimum value of f is 6.

- $L_3 = \{(2, y) \mid 0 \leq y \leq 3\}$: on L_3, the function $f(x, y)$ becomes

$$f(2, y) = 4 + 2y^2 - 4 - 4y + 1 = 2y^2 - 4y + 1$$

Finding the maximum and the minimum values of f on L_3 amounts to finding the global maximum and minimum values of $f(y) = 2y^2 - 4y + 1$ on $0 \leq y \leq 3$. This situation is identical to the situation over L_1, so on L_3, the maximum value of f is 7, and the minimum value of f is -1.

- $L_4 = \{(x, 0) \mid 0 \leq x \leq 2\}$: on L_4, the function $f(x, y)$ becomes

$$f(x, 0) = x^2 - 2x + 1$$

Finding the maximum and the minimum values of f on L_4 amounts to finding the global maximum and minimum values of $f(x) = x^2 - 2x + 1$ on $0 \leq x \leq 2$. This function has the critical point when $f'(x) = 2x - 2 = 0$, or $x = 1$. At this point, the value of f is $f(1) = 1 - 2 + 1 = 0$. There are two boundary points, $x = 0$ and $x = 2$, at which the values of f are $f(0) = 1$ and $f(2) = 4 - 4 + 1 = 1$. Thus, on L_4, the maximum value of f is 1, and the minimum value of f is 0.

Combining all of these, we find that the global maximum value of $f(x, y)$ on D is 7, and the global minimum value of $f(x, y)$ on D is -2.

□

Exercise 2. Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$.

Solution. If (x, y, z) is on the plane $x + y + z = 1$, the value of z is expressed in terms of the values of x, y via $z = 1 - x - y$. The question is equivalent to asking the global minimum value of

$$f(x, y) = \sqrt{(2 - x)^2 + (0 - y)^2 + (-3 - (1 - x - y))^2}$$

Since the distance from a point on the plane $x+y+z = 1$ to $(2, 0, -3)$ grows larger as the point veers off towards infinity, there is no boundary point to be considered. Thus, we just need to find the critical points of $f(x, y)$ and take the minimum values of f on the critical points.

The function $f(x, y)$ is

$$f(x, y) = \sqrt{(2 - x)^2 + y^2 + (x + y - 4)^2} = \sqrt{(x^2 - 4x + 4) + y^2 + (x^2 + y^2 + 16 - 8x - 8y + 2xy)}$$
$$= \sqrt{2x^2 + 2xy + 2y^2 - 12x - 8y + 20}$$

The point (x, y) is a critical point if

$$f_x(x, y) = \frac{4x + 2y - 12}{2\sqrt{2x^2 + 2xy + 2y^2 - 12x - 8y + 20}} = \frac{2x + y - 6}{\sqrt{2x^2 + 2xy + 2y^2 - 12x - 8y + 20}} = 0$$

and

$$f_y(x, y) = \frac{2x + 4y - 8}{2\sqrt{2x^2 + 2xy + 2y^2 - 12x - 8y + 20}} = \frac{x + 2y - 4}{\sqrt{2x^2 + 2xy + 2y^2 - 12x - 8y + 20}} = 0$$

This means that $2x + y - 6 = 0$ and $x + 2y - 4 = 0$, or

$$2x + y = 6, \quad x + 2y = 4$$

The second equation means $x = 4 - 2y$, and we can plug this into the first equation:

$$2(4 - 2y) + y = 6,$$

or $2 - 3y = 0$, or $y = \frac{2}{3}$. From this, $x = 4 - \frac{4}{3} = \frac{8}{3}$. The shortest distance is thus $f(\frac{8}{3}, \frac{2}{3})$, given by

$$f(\frac{8}{3}, \frac{2}{3}) = \sqrt{2 \cdot \frac{64}{9} + 2 \cdot \frac{16}{9} + 2 \cdot \frac{4}{9} - 12 \cdot \frac{8}{3} - 8 \cdot \frac{2}{3} + 20}$$
$$= \sqrt{\frac{168}{9} - \frac{112}{3} + 20}$$
$$= \sqrt{\frac{56}{3} - \frac{112}{3} + 20} = \sqrt{-\frac{56}{3} + 20} = \sqrt{\frac{4}{3}}$$

□

Exercise 3. Find the point on the plane $x - 2y + 3z = 6$ that is closest to the point $(0, 1, 1)$.

Solution. If (x, y, z) is on the plane $x - 2y + 3z = 6$, the value of x is expressed in terms of the values of y, z via $x = 2y - 3z + 6$. The question is equivalent to asking the point that the function

$$f(y, z) = \sqrt{(0 - (2y - 3z + 6))^2 + (1 - y)^2 + (1 - z)^2}$$

achieves its global minimum. Since the distance from a point on the plane $x - 2y + 3z = 6$ to $(0, 1, 1)$ grows infinitely larger as the point veers off towards infinity, there is no boundary point to be considered. Thus, we just need to find the critical points of $f(y, z)$ and take the minimum values of f on the critical points.

The function $f(y, z)$ is

$$f(y, z) = \sqrt{(0 - (2y - 3z + 6))^2 + (1 - y)^2 + (1 - z)^2}$$

$$= \sqrt{(4y^2 + 9z^2 + 36 - 12yz + 24y - 36z) + (y^2 - 2y + 1) + (z^2 - 2z + 1)}$$

$$= \sqrt{5y^2 + 10z^2 - 12yz + 22y - 38z + 38}$$

The point (y, z) is a critical point if

$$f_y(y, z) = \frac{10y - 12z + 22}{2\sqrt{5y^2 + 10z^2 - 12yz + 22y - 38z + 38}} = \frac{5y - 6z + 11}{\sqrt{5y^2 + 10z^2 - 12yz + 22y - 38z + 38}} = 0$$

and

$$f_z(y, z) = \frac{20z - 12y - 38}{2\sqrt{5y^2 + 10z^2 - 12yz + 22y - 38z + 38}} = \frac{10z - 6y - 19}{\sqrt{5y^2 + 10z^2 - 12yz + 22y - 38z + 38}} = 0$$

This means that $5y - 6z + 11 = 0$ and $10z - 6y - 19 = 0$. The second equation means $z = \frac{3}{5}y + \frac{19}{10}$, which can be plugged into the first equation to obtain

$$5y - 6(\frac{3}{5}y + \frac{19}{10}) + 11 = 0,$$

or

$$\frac{7}{5}y - \frac{2}{5} = 0,$$

or $y = \frac{2}{7}$. From this, $z = \frac{3}{5} \cdot \frac{2}{7} + \frac{19}{10} = \frac{29}{14}$, and $x = 2 \cdot \frac{2}{7} - 3 \cdot \frac{29}{14} + 6 = \frac{5}{14}$. Thus, the closest point is $(\frac{5}{14}, \frac{2}{7}, \frac{29}{14})$. $\square$

Exercise 4. Find the point on the surface $z = x^2 + y^2$ that are closest to the point $(5, 5, 0)$.

Solution. If (x, y, z) is on the surface, the value of z is expressed in terms of the values of x, y via $z = x^2 + y^2$. The question is equivalent to asking the point that the global minimum value of

$$f(x, y) = \sqrt{(x - 5)^2 + (y - 5)^2 + (x^2 + y^2 - 0)^2}$$

is achieved. If x or y goes to infinity or negative infinity, this function goes to infinity, so there is no need to worry about the boundary. Thus, we just need to find the critical points of $f(x, y)$ and take the minimum.

The function $f(x, y)$ is

$$f(x, y) = \sqrt{(x - 5)^2 + (y - 5)^2 + (x^2 + y^2 - 0)^2} = \sqrt{(x^2 - 10x + 25) + (y^2 - 10y + 25) + (x^4 + 2x^2y^2 + y^4}$$

$$= \sqrt{x^4 + 2x^2y^2 + y^4 + x^2 + y^2 - 10x - 10y + 50}$$

The point (x, y) is a critical point if

$$f_x(x,y) = \frac{4x^3 + 4xy^2 + 2x - 10}{2\sqrt{x^4 + 2x^2y^2 + y^4 + x^2 + y^2 - 10x - 10y + 50}}$$
$$= \frac{2x^3 + 2xy^2 + x - 5}{\sqrt{x^4 + 2x^2y^2 + y^4 + x^2 + y^2 - 10x - 10y + 50}} = 0$$

and

$$f_y(x,y) = \frac{4x^2y + 4y^3 + 2y - 10}{2\sqrt{x^4 + 2x^2y^2 + y^4 + x^2 + y^2 - 10x - 10y + 50}}$$
$$= \frac{2x^2y + 2y^3 + y - 5}{\sqrt{x^4 + 2x^2y^2 + y^4 + x^2 + y^2 - 10x - 10y + 50}} = 0$$

This means

$$2x^3 + 2xy^2 + x - 5 = 0, \quad 2x^2y + 2y^3 + y - 5 = 0$$

Subtracting the second equation from the first equation, we get

$$(2x^3 + 2xy^2 + x - 5) - (2x^2y + 2y^3 + y - 5) = 0,$$

or

$$2x^3 - 2x^2y + 2xy^2 - 2y^3 + x - y = 0$$

,or

$$(x - y)(2x^2 + 2y^2 + 1) = 0$$

Since $2x^2 + 2y^2 + 1 \geq 1$ is never zero, this means $x - y = 0$, or $x = y$. Using this, the first equation becomes

$$2x^3 + 2x^3 + x - 5 = 0,$$

or $4x^3 + x - 5 = 0$. This can be factorized as $(x-1)(4x^2+4x+5) = 0$. Note that $4x^2+4x+5 = 0$ has no roots (alternatively, $4x^2 + 4x + 5 = (2x + 1)^2 + 4 \geq 4$ is never zero), so the critcial points can happen only if $x = y = 1$. Then $z = 2$. This point, $(1, 1, 2)$, is thus the point that is closest to $(5, 5, 0)$. □

Exercise 5. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

Solution. If (x, y, z) is on the surface, then either $y = \sqrt{9 + xz}$ or $y = -\sqrt{9 + xz}$. Since the distance from (x, y, z) to $(0, 0, 0)$ is the same as the distance from $(x, -y, z)$ to $(0, 0, 0)$, you only need to consider $y = \sqrt{9 + xz}$ to find the closest distance. The problem of finding the closest distance to the origin becomes the problem of finding the global minimum value of

$$f(x, z) = \sqrt{x^2 + (9 + xz) + z^2} = \sqrt{x^2 + xz + z^2 + 9}$$

As before, the global minimum does not show up as x, z go to infinity. The critical points happen when

$$f_x(x,z) = \frac{2x + z}{2\sqrt{x^2 + xz + z^2 + 9}} = 0,$$
$$f_z(x,z) = \frac{x + 2z}{2\sqrt{x^2 + xz + z^2 + 9}} = 0$$

so this means

$$2x + z = 0, \quad x + 2z = 0$$

The second equation is $x = -2z$, so plugging this into the first equation, we get $2(-2z) + z = 0$, or $-3z = 0$, or $z = 0$, so $x = 0$. So the closest distance is $f(0, 0) = 9$ and happens at $(0, 3, 0)$. Since the same distance is achieved at $(0, -3, 0)$, the points on the surface closest to the origin are $(0, 3, 0)$ and $(0, -3, 0)$. □

18. Lagrange multipliers

Exercise 1. Find the global maximum and minimum values of f subject to the given constraint.

(1) $f(x, y) = x^2 - y^2$, on $x^2 + y^2 = 1$
(2) $f(x, y) = xe^y$, on $x^2 + y^2 = 2$
(3) $f(x, y) = xye^{-x^2-y^2}$, on $x^2 + y^2 = 1$
(4) $f(x, y, z) = xy^2z$, on $x^2 + y^2 + z^2 = 4$
(5) $f(x, y, z) = x^2 + y^2 + z^2$, on $x^2 + y^2 + z^2 + xy = 12$
(6) $f(x, y, z) = x^4 + y^4 + z^4$, on $x^2 + y^2 + z^2 = 1$

Solution.

(1) The constraint is $g(x, y) = 1$ where $g(x, y) = x^2 + y^2$. So the global max/min can occur at the Lagrange critical points, namely when $\nabla f(x, y)$ and $\nabla g(x, y)$ are parallel. Since $\nabla f(x, y) = \langle 2x, -2y\rangle$ and $\nabla g(x, y) = \langle 2x, 2y\rangle$, this happens either when $\nabla g(x, y) = \langle 0, 0\rangle$, which is when $x = y = 0$, which contradicts $x^2 + y^2 = 1$, or there is λ such that $\langle 2x, -2y\rangle = \lambda\langle 2x, 2y\rangle$. Since $2x = 2\lambda x$, either $\lambda = 1$ or $x = 0$.
   - If $\lambda = 1$, then $-2y = 2y$, so $y = 0$. Then $x^2 = 1$, so $f(x, y) = 1$.
   - If $x = 0$, then $y^2 = 1$, so $f(x, y) = -1$.

   So the global max is 1 and the global min is -1.
(2) The constraint is $g(x, y) = 2$ where $g(x, y) = x^2 + y^2$. So the global max/min can occur at the Lagrange critical points, namely when $\nabla f(x, y)$ and $\nabla g(x, y)$ are parallel. Since $\nabla f(x, y) = \langle e^y, xe^y\rangle$ and $\nabla g(x, y) = \langle 2x, 2y\rangle$, This happens either when $\nabla g(x, y) = \langle 0, 0\rangle$, which is when $x = y = 0$, which contradicts $x^2 + y^2 = 2$, or there is λ such that $\langle e^y, xe^y\rangle = \lambda\langle 2x, 2y\rangle$. So $2\lambda x^2 = 2\lambda y$, so either $\lambda = 0$ or $x^2 = y$.
   - If $\lambda = 0$, then $e^y = 0$, which is a contradiction.
   - If $x^2 = y$, then $x^2 + y^2 = 2$ becomes $x^4 + x^2 - 2 = 0$. This factorizes into $(x^2 - 1)(x^2 + 2) = 0$. So either $x^2 = 1$ or $x^2 = -2$. Since x^2 is positive, $x^2 = 1$, so either $x = 1$ or $x = -1$. Thus $y = 1$.

   So the Lagrange critical poitns are $(1, 1)$ and $(-1, 1)$. Since $f(1, 1) = e$ and $f(-1, 1) = -e$, the global max is e and the global min is $-e$.
(3) The constraint is $g(x, y) = 1$ where $g(x, y) = x^2 + y^2$. So the global max/min can occur at the Lagrange critical points, namely when $\nabla f(x, y)$ and $\nabla g(x, y)$ are parallel. Since $\nabla f(x, y) = \langle ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2}, xe^{-x^2-y^2} - 2xy^2e^{-x^2-y^2}\rangle$ and $\nabla g(x, y) = \langle 2x, 2y\rangle$, they can be parallel if there is λ such that

$$(1 - 2x^2)ye^{-x^2-y^2} = 2\lambda x, \quad (1 - 2y^2)xe^{-x^2-y^2} = 2\lambda y.$$

   So

$$(1 - 2x^2)y^2e^{-x^2-y^2} = 2\lambda xy = (1 - 2y^2)x^2e^{-x^2-y^2}$$

   so

$$(1 - 2x^2)y^2 = (1 - 2y^2)x^2$$

Expanding out we get

$$y^2 - 2x^2y^2 = x^2 - 2x^2y^2$$

or $y^2 = x^2$. So either $x = y$ or $x = -y$. Using $x^2 + y^2 = 1$, we get $2x^2 = 1$, or $x = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$. The function $x^2 + y^2$ is set to be 1, and xy is maximized when $x = y$ which is $1/2$ and minimized when $x = -y$ which is $-1/2$. So the global maximum is $\frac{e^{-1}}{2}$ and the global minimum is $-\frac{e^{-1}}{2}$.

(4) The constraint is $g(x, y, z) = 4$ where $g(x, y, z) = x^2 + y^2 + z^2$.

$$\nabla f(x, y, z) = \langle y^2 z, 2xyz, xy^2 \rangle, \quad \nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$$

In order for them to be parallel, either ∇g is zero or there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

- If $\nabla g(x, y, z) = \langle 0, 0, 0 \rangle$, then $x = y = z = 0$, which cannot happen as $x^2 + y^2 + z^2 = 4$.
- If there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, then

  $$y^2 z = 2\lambda x, \quad 2xyz = 2\lambda y, \quad xy^2 = 2\lambda z$$

  So

  $$xy^2 z = 2\lambda x^2, \quad xy^2 z = \lambda y^2, \quad xy^2 z = 2\lambda z^2$$

  so

  $$2\lambda x^2 = \lambda y^2 = 2\lambda z^2$$

  so either $\lambda = 0$ or $2x^2 = y^2 = 2z^2$.
  - If $\lambda = 0$, then $y^2 z = 0$, which means either $y = 0$ or $z = 0$. In either case, $f(x, y, z) = 0$.
  - If $2x^2 = y^2 = 2z^2$, we use this with $x^2 + y^2 = z^2 = 4$ to get

    $$x^2 + 2x^2 + x^2 = 4,$$

    or $x^2 = 1$, or $x = 1$ or $x = -1$. Then $y^2 = 2$, and $z^2 = 1$, so $z = 1$ or $z = -1$. Thus $f(x, y, z)$ is either 2 or -2.

Combining all these, the global maximum is 2 and the global minimum is -2.

(5) The constratint is $g(x, y, z) = 12$ where $g(x, y, z) = x^2 + y^2 + z^2 + xy$.

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle, \quad \nabla g(x, y, z) = \langle 2x + y, 2y + x, 2z \rangle$$

For them to be parallel, either $\nabla g(x, y, z)$ is zero or there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.

- If $\nabla g(x, y, z) = \langle 0, 0, 0 \rangle$, then $z = 0$, and $2x + y = 0$ and $2y + x = 0$. This solves into $x - y = 0$, so $x = y$, so $x = y = 0$. This in turn is impossible as $x^2 + y^2 + z^2 + xy = 12$.
- If there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, we have

  $$2x = \lambda(2x + y), \quad 2y = \lambda(2y + x), \quad 2z = 2\lambda z$$

  From the third equation, either $\lambda = 1$ or $z = 0$.
  - If $\lambda = 1$, we have

    $$2x = 2x + y, \quad 2y = 2y + x$$

    so $x = 0$ and $y = 0$. The constraint then becomes $z^2 = 12$, at which $f(x, y, z) = 12$.

– If $z = 0$, we have

$$2x = \lambda(2x + y), \quad 2y = \lambda(2y + x)$$

Adding these two, we get

$$2(x + y) = 3\lambda(x + y)$$

so either $x + y = 0$ or $2 = 3\lambda$.

∗ If $x + y = 0$, or $y = -x$, we get $2x = \lambda x$, so either $\lambda = 0$ or $x = 0$.

· If $\lambda = 0$, then this means $x = y = z = 0$, which is not allowed.

· If $x = 0$, then $y = 0$, and we already had $z = 0$, so $x = y = z = 0$ which is not allowed.

∗ If $\lambda = \frac{2}{3}$, then $2x = \frac{2}{3}(2x + y)$ implies $6x = 4x + 2y$, or $2x = 2y$, or $x = y$. Putting this into the constraint, we get $3x^2 = 12$, or $x^2 = 4$. So $f(x, y, z) = 8$.

Combining all these, we get 8 is the global minimum, and 12 is the global maximum.

(6) The constraint is $g(x, y, z) = 1$ where $g(x, y, z) = x^2 + y^2 + z^2$.

$$\nabla f(x, y, z) = \langle 4x^3, 4y^3, 4z^3\rangle, \quad \nabla g(x, y, z) = \langle 2x, 2y, 2z\rangle$$

For them to be parallel, either $\nabla g(x, y, z)$ is zero or there is λ such that $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)$.

- If $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$, $x = y = z = 0$, which is not allowed as $x^2 + y^2 + z^2 = 1$.
- If there is λ such that $\nabla f(x, y, z) = \lambda\nabla g(x, y, z)$, then

$$4x^3 = 2\lambda x, \quad 4y^3 = 2\lambda y, \quad 4z^3 = 2\lambda z$$

So from the first equation, either $x = 0$ or $2x^2 = \lambda$.

– If $x = 0$, the second equation says either $y = 0$ or $2y^2 = \lambda$.

∗ If $y = 0$, then $x = y = 0$, so $z^2 = 1$, which means $f(x, y, z) = 1$.

∗ If $2y^2 = \lambda$, the third equation says either $z = 0$ or $2z^2 = \lambda$.

· If $z = 0$, then $x = z = 0$, so $y^2 = 1$, which means $f(x, y, z) = 1$.

· If $2z^2 = \lambda$, then $2y^2 = 2z^2$, so $x^2 = 0$, $y^2 = z^2 = \frac{1}{2}$. So $f(x, y, z) = \frac{1}{2}$.

– If $2x^2 = \lambda$, the second equation says either $y = 0$ or $2y^2 = \lambda$.

∗ If $y = 0$, then the third equation says either $z = 0$ or $2z^2 = \lambda$.

· If $z = 0$, then $y = z = 0$, so $x^2 = 1$, so $f(x, y, z) = 1$.

· If $2z^2 = \lambda$, then $2x^2 = 2z^2$, so $x^2 = z^2 = \frac{1}{2}$ while $y^2 = 0$, so $f(x, y, z) = \frac{1}{2}$.

∗ If $2y^2 = \lambda$, the third equation says either $z = 0$ or $2z^2 = \lambda$.

· If $z = 0$, then $2x^2 = 2y^2$ while $z^2 = 0$, so $x^2 = y^2 = \frac{1}{2}$, so $f(x, y, z) = \frac{1}{2}$.

· If $2z^2 = \lambda$, then $2x^2 = 2y^2 = 2z^2$, so $x^2 = y^2 = z^2 = \frac{1}{3}$. So $f(x, y, z) = \frac{1}{3}$.

Combining these, the global minimum is $\frac{1}{3}$, and the global maximum is 1.

□

Exercise 2. Find the global maximum and minimum values of f on the given region.

(1) $f(x, y) = x^2 + y^2 + 4x - 4y$, on $x^2 + y^2 \leq 9$
(2) $f(x, y) = 2x^2 + 3y^2 - 4x - 5$, on $x^2 + y^2 \leq 16$
(3) $f(x, y) = \sin(x + y)$, on $x^2 + xy + y^2 \leq 3$

(4) $f(x,y,z) = xyz$, on $x^2+y^2+z^2 \le 1$
(5) $f(x,y,z) = x^2+y^2+z^2$, on $x^4+y^4+z^4 \le 1$
(6) $f(x,y,z) = x^2+y^2+z^2$, on $x^2+y^2+z^2+xy-xz-yz \le 1$

Solution.

(1) Use the 4-step process.
- Step 1: Find the critical points. Note that
$$\nabla f(x,y) = \langle 2x+4, 2y-4\rangle$$
so $\nabla f(x,y) = \langle 0,0\rangle$ means $2x+4=0$ and $2y-4=0$, or $x=-2$ and $y=2$. Since $(-2,2)$ does belong to the region $x^2+y^2 \le 9$, $(-2,2)$ is a critical point.
- Step 2: Find the max/min values of f at the critical points. Both the max and min values of f are $f(-2,2) = 4+4-8-8 = -8$.
- Step 3: Find the global max/min of f on the boundary. On the boundary we have a constraint $g(x,y) = 9$, where $g(x,y) = x^2+y^2$. By the method of Lagrange multipliers, we would like to find a point (x,y) where $\nabla f(x,y) = \langle 2x+4, 2y-4$ is parallel to $\nabla g(x,y) = \langle 2x, 2y\rangle$. This can happen either when $\nabla g(x,y) = \langle 0,0\rangle$ or there is λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.
  - If $\nabla g(x,y) = \langle 0,0\rangle$, this means $x=y=0$, which does not satisfy the constraint $x^2+y^2=9$.
  - Suppose there is λ such that $\langle 2x+4, 2y-4\rangle = \lambda\langle 2x, 2y\rangle$. Then
  $$2x+4 = 2\lambda x, \quad 2y-4 = 2\lambda y$$
  or
  $$2 = (\lambda-1)x, \quad -2 = (\lambda-1)y$$
  so $(\lambda-1)x = -(\lambda-1)y$. Thus, either $\lambda=1$ or $x=-y$.
    * If $\lambda = 1$, then $2 = (\lambda-1)x$ implies $2=0$, so this doesn't make sense.
    * If $x=-y$, then $x^2+y^2=9$ implies that $2x^2=9$, or $x = \frac{3}{\sqrt{2}}$ or $-\frac{3}{\sqrt{2}}$. Thus the Lagrange critical points are $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ and $(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$.

  Thus the Lagrange critical points are $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ and $(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$, and over them the values of f are $f(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}) = 9 + \frac{12}{\sqrt{2}} + \frac{12}{\sqrt{2}} = 9+12\sqrt{2}$ and $f(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}) = 9 - \frac{12}{\sqrt{2}} - \frac{12}{\sqrt{2}} = 9-12\sqrt{2}$.
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. The maximum will be $9+12\sqrt{2}$ and the minimum will be -8.

(2) Use the 4-step process.
- Step 1: Find the critical points. Note
$$\nabla f(x,y) = \langle 4x-4, 6y\rangle$$
so if (x,y) is a critical point, $4x-4=0$ and $6y=0$, or $x=1$ and $y=0$. The point $(1,0)$ appears in the region $x^2+y^2 \le 16$, so is a critical point.
- Step 2: Find the max/min values of f at the critical points. This would be $f(1,0) = 2-4-5 = -7$.
- Step 3: Find the global max/min of f on the boundary. On the boundary, we have a new constraint $g(x,y) = 16$ where $g(x,y) = x^2+y^2$. By the method of Lagrange multipliers, the Lagrange critical points happen when $\nabla f(x,y) = \langle 4x-4, 6y\rangle$ is

parallel to $\nabla g(x,y) = \langle 2x, 2y\rangle$. This happens either when $\nabla g(x,y)$ is zero or when there is a number λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.

  - If $\nabla g(x,y) = \langle 0,0\rangle$, this means $x = y = 0$, which contradicts the constraint $x^2 + y^2 = 16$.
  - Suppose there is a number λ such that $\langle 4x-4, 6y\rangle = \lambda\langle 2x, 2y\rangle$. This means that

$$4x - 4 = 2\lambda x, \quad 6y = 2\lambda y,$$

    so from the second equation, either $\lambda = 3$ or $y = 0$.
    * If $\lambda = 3$, then the first equation says $4x - 4 = 6x$, or $2x = -4$, or $x = -2$. From this, $x^2 + y^2 = 16$ becomes $y^2 = 12$, so $y = \sqrt{12}$ or $-\sqrt{12}$. So we obtain two Lagrange critical points $(-2, \sqrt{12})$ and $(-2, -\sqrt{12})$.
    * If $y = 0$, then $x^2 + y^2 = 16$ becomes $x^2 = 16$, so $x = 4$ or -4. So we obtain two Lagrange critical points $(4,0)$ and $(-4,0)$.

  At the four Lagrange critical points, $(-2,\sqrt{12})$, $(-2,-\sqrt{12})$, $(4,0)$ and $(-4,0)$, the values of f are

$$f(-2,\sqrt{12}) = 8 + 36 + 8 - 5 = 47$$
$$f(-2,-\sqrt{12}) = 8 + 36 + 8 - 5 = 47$$
$$f(4,0) = 32 - 16 - 5 = 11$$
$$f(-4,0) = 32 + 16 - 5 = 43$$

  So the global maximum value on the boundary is 47 and the global minimum value on the 11.
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. The global maximum is 47 and the global minimum is -7.

(3) Use the 4-step process.

- Step 1: Find the critical points. We have

$$\nabla f(x,y) = \langle \cos(x+y), \cos(x+y)\rangle$$

  so the critical points happen when $\cos(x+y) = 0$. This happens when $x+y$ is an odd integer times $\frac{\pi}{2}$ (such as $\frac{\pi}{2}$, $\frac{3\pi}{2}$, $-\frac{\pi}{2}$).
- Step 2: Find the max/min values of f at the critical points. We have $x+y$ equal to an odd integer times $\frac{\pi}{2}$, so $\sin(x+y)$ is either 1 or -1. So the maximum value on the critical points is 1 and the minimum value on the critical points is -1.
- Step 3: Find the global max/min of f on the boundary. On the boundary we have another constraint $g(x,y) = 3$ where $g(x,y) = x^2 + xy + y^2$. Lagrange critical points are when $\nabla f(x,y) = \langle \cos(x+y), \cos(x+y)\rangle$ is parallel to $\nabla g(x,y) = \langle 2x+y, x+2y\rangle$. This happens when either $\nabla g(x,y)$ is zero or there is a number λ such that $\nabla f(x,y) = \lambda\nabla g(x,y)$.
  - If $\nabla g(x,y) = \langle 0,0\rangle$, this means $2x+y = 0$ and $x+2y = 0$. Subtracting the second equation from the first equation, we get $x - y = 0$, or $x = y$. Plugging this back into $2x + y = 0$, we get $3x = 0$, or $x = 0$. So $x = y = 0$. This conflicts with the constraint $x^2 + xy + y^2 = 3$.
  - If there is a number λ such that $\langle \cos(x+y), \cos(x+y)\rangle = \lambda\langle 2x+y, x+2y\rangle$, this means $\lambda(2x+y) = \lambda(x+2y)$, or $\lambda(x-y) = 0$. So either $\lambda = 0$ or $x = y$.

* If $\lambda = 0$, then $\cos(x+y) = 0$, so $\nabla f(x,y) = \langle 0, 0\rangle$, so this case is analyzed already.
* If $x = y$, then the constraint $x^2 + xy + y^2 = 3$ becomes $3x^2 = 3$, or $x^2 = 1$, so $x = 1$ or $x = -1$. On $(1,1)$ and $(-1,-1)$, the values of f are $\sin(2)$ and $\sin(-2)$.

- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. So the global maximum value of f is 1, and the global minimum value of f is -1.

(4) Use the 4-step process.

- Step 1: Find the critical points. Note
$$\nabla f(x,y,z) = \langle yz, xz, xy\rangle,$$
so this is zero if $yz = 0$, $xz = 0$, and $xy = 0$. The first equation means either $y = 0$ or $z = 0$. Either way, $f(x,y,z) = xyz$ is 0 on a critical point.
- Step 2: Find the max/min values of f at the critical points. This is already dealt above.
- Step 3: Find the global max/min of f on the boundary. On the boundary, we have a constraint $g(x,y,z) = 1$, where $g(x,y,z) = x^2+y^2+z^2$. Lagrange critical points happen when $\nabla f(x,y,z) = \langle yz, xz, xy\rangle$ are parallel to $\nabla g(x,y,z) = \langle 2x, 2y, 2z\rangle$. This happens either when $\nabla g(x,y,z)$ is zero or there is a number λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$.
  - If $\langle 2x, 2y, 2z\rangle = \langle 0,0,0\rangle$, $x = y = z = 0$. This contradicts with the constraint $x^2 + y^2 + z^2 = 1$.
  - If there is a number λ such that $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$, we have
  $$yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z$$
  So
  $$xyz = 2\lambda x^2, \quad xyz = 2\lambda y^2, \quad xyz = 2\lambda z^2$$
  so $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$. So either $\lambda = 0$ or $x^2 = y^2 = z^2$.
    * If $\lambda = 0$, then $\nabla f(x,y,z) = \langle 0,0,0\rangle$, so this is already deal with.
    * If $x^2 = y^2 = z^2$, then from $x^2 + y^2 + z^2 = 1$, we have x, y, z equal to either $\frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$. So xyz is either $\frac{1}{3\sqrt{3}}$ or $-\frac{1}{3\sqrt{3}}$.
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. The global maximum value is $\frac{1}{3\sqrt{3}}$, and the global minimum value is $-\frac{1}{3\sqrt{3}}$.

(5) Use the 4-step process.

- Step 1: Find the critical points. Note
$$\nabla f(x,y,z) = \langle 2x, 2y, 2z\rangle$$
So the critical point is $(0,0,0)$.
- Step 2: Find the max/min values of f at the critical points. On the critical point, $f(0,0,0) = 0$.
- Step 3: Find the global max/min of f on the boundary. On the boundary we have an extra constraint $g(x,y,z) = 1$ where $g(x,y,z) = x^4 + y^4 + z^4$. Lagrange critical points happen when $\nabla f(x,y,z)$ is parallel to $\nabla g(x,y,z) = \langle 4x^3, 4y^3, 4z^3\rangle$. This happens either when $\nabla g(x,y,z)$ is zero or when there is λ such that $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)$.
  - If $\nabla g(x,y,z) = \langle 0,0,0\rangle$, then $x = y = z = 0$, which contradicts the constraint $x^4 + y^4 + z^4 = 1$.

- If there is λ such that $\langle 2x, 2y, 2z\rangle = \lambda\langle 4x^3, 4y^3, 4z^3\rangle$, we have

$$2x = 4\lambda x^3, \quad 2y = 4\lambda y^3, \quad 2z = 4\lambda z^3$$

  From the first equation, either $x = 0$ or $2 = 4\lambda x^2$.
  * If $x = 0$, the second equation tells either $y = 0$ or $2 = 4\lambda y^2$.
    · If $y = 0$, then $x = y = 0$ implies $z^4 = 1$, so $z^2 = 1$. So the value of f is 1.
    · If $2 = 4\lambda y^2$, then the third equation tells either $z = 0$ or $2 = 4\lambda z^2$.
    If $z = 0$, then $x = z = 0$ implies $y^4 = 1$, so $y^2 = 1$. So the value of f is 1.
    If $2 = 4\lambda z^2$, then $y^2 = \frac{1}{2\lambda} = z^2$ while $x = 0$, so $2y^4 = 1$ or $y^4 = \frac{1}{2}$, so $y^2 = \frac{1}{\sqrt{2}} = z^2$, so the value of f is $\frac{2}{\sqrt{2}} = \sqrt{2}$.
  * If $2 = 4\lambda x^2$, then $x^2 = \frac{1}{2\lambda}$. The second equation tells either $y = 0$ or $2 = 4\lambda y^2$.
    · If $y = 0$, then the third equation tells either $z = 0$ or $2 = 4\lambda z^2$.
    If $z = 0$, then $y = z = 0$ implies that $x^4 = 1$, or $x^2 = 1$. So the value of f is 1.
    If $2 = 4\lambda z^2$, then $z^2 = \frac{1}{2\lambda}$. So $x^2 = z^2$ while $y = 0$, so $2x^4 = 1$, or $x^4 = \frac{1}{2}$, or $x^2 = \frac{1}{\sqrt{2}} = z^2$. So the value of f is $\sqrt{2}$.
  * If $2 = 4\lambda y^2$, then $y^2 = \frac{1}{2\lambda} = x^2$. The third equation tells either $z = 0$ or $2 = 4\lambda z^2$.
    · If $z = 0$, then $2x^4 = 1$, or $x^4 = \frac{1}{2}$, or $x^2 = \frac{1}{\sqrt{2}} = y^2$, so the value of f is $\sqrt{2}$.
    · If $2 = 4\lambda z^2$, then $z^2 = \frac{1}{2\lambda} = x^2 = y^2$, so $3x^4 = 1$, or $x^4 = \frac{1}{3}$, or $x^2 = \frac{1}{\sqrt{3}}$, so the value of f is $\sqrt{3}$.
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. The global maximum value is $\sqrt{3}$, and the global minimum value is 0.

(6) Use the 4-step process.
- Step 1: Find the critical points. Note

$$\nabla f(x, y, z) = \langle 2x, 2y, 2z\rangle$$

  so the critical point is $(0, 0, 0)$.
- Step 2: Find the max/min values of f at the critical points. On the critical point $f(0, 0, 0) = 0$.
- Step 3: Find the global max/min of f on the boundary. On the boudnary there is a constraint $g(x, y, z) = 1$ where $g(x, y, z) = x^2 + y^2 + z^2 + xy - xz - yz$. Lagrange critical points happen when $\nabla f(x, y, z)$ is parallel to $\nabla g(x, y, z) = \langle 2x + y - z, 2y + x - z, 2z - x - y\rangle$. This happens when either $\nabla g(x, y, z)$ is zero or there is a number λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
  - If $\nabla g(x, y, z) = \langle 0, 0, 0\rangle$, we have

$$2x + y - z = 0, \quad 2y + x - z = 0, \quad 2z - x - y = 0$$

  If you add all three, we get $2x + 2y = 0$, or $x + y = 0$. So $2z = 0$, so $z = 0$. So $2x + y = x = 0$, so $x = 0$, and $y = 0$. But $x = y = z = 0$ contradicts the constraint $x^2 + y^2 + z^2 + xy - xz - yz = 0$.
  - If there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ we have

$$2x = \lambda(2x + y - z), \quad 2y = \lambda(2y + x - z), \quad 2z = \lambda(2z - x - y)$$

If you subtract the second equation from the first equation, you get

$$2(x-y)=\lambda(x-y),$$

so either $x-y=0$ or $\lambda=2$.

$*$ If $x-y=0$, then $x=y$, so we have

$$2x=\lambda(3x-z), \quad z=\lambda(z-x)$$

If you add them you get

$$2x+z=2\lambda x$$

so $z=2(\lambda-1)x$. On the other hand, the second equation tells you $\lambda x=(\lambda-1)z$, so

$$\lambda x=(\lambda-1)z=2(\lambda-1)^2x=(2\lambda^2-4\lambda+2)x$$

so either $x=0$ or $\lambda=2\lambda^2-4\lambda+2$.

$\cdot$ If $x=0$, then $y=0$, so the constraint becomes $z^2=1$, so the value of f is 1.

$\cdot$ If $\lambda=2\lambda^2-4\lambda+2$, then $2\lambda^2-5\lambda+2=0$, so $(2\lambda-1)(\lambda-2)=0$, so either $\lambda=\frac{1}{2}$ or $\lambda=2$.
If $\lambda=\frac{1}{2}$, then we have

$$2x=\frac{1}{2}(3x-z), \quad z=\frac{1}{2}(z-x)$$

or

$$4x=3x-z, \quad 2z=z-x$$

or $z=-x$. So $x=y=-z$. The constraint becomes $6x^2=1$, or $x^2=\frac{1}{6}$, so the value of f is $\frac{1}{2}$.
If $\lambda=2$, then we have

$$2x=2(3x-z), \quad z=2(z-x),$$

or

$$2x=6x-2z, \quad z=2z-2x$$

or $2x=z$. The constraint becomes $3x^2=1$, or $x^2=\frac{1}{3}$. So the value of f is 2.

$*$ If $\lambda=2$, then we have

$$x=2x+y-z, \quad y=2y+x-z, \quad z=2z-x-y,$$

so $x+y=z$. The constraint becomes

$$x^2+y^2+(x+y)^2+xy-(x+y)^2=1,$$

or $x^2+xy+y^2=1$. The value of f is then $x^2+y^2+(x+y)^2=2x^2+2xy+2y^2=2$.

- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. So the global maximum is 2 and the global minimum is 0.

□

Exercise 3. Find the global maximum and minimum values of f subject to the given constraint.

(1) $f(x,y)=x^2y$, on $x^2+y^2=1$, $y\geq 0$.

(2) $f(x,y) = e^{-x^2-y^2}(x^2+2y^2)$, on $x^2+y^2=4$, $x+y \geq 0$.
(3) $f(x,y,z) = xyz$, on $x^2+y^2+z^2=3$, $z \geq 0$.

Solution.

(1) Use the 4-step process.
- Step 1: Find the Lagrange critical points. This happens when $\nabla f(x,y) = \langle 2xy, x^2 \rangle$ is parallel to $\nabla g(x,y)$, where $g(x,y) = x^2+y^2$. So $\nabla g(x,y) = \langle 2x, 2y \rangle$. This happens when either $\nabla g(x,y)$ is zero or there is a number λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.
  - If $\nabla g(x,y) = \langle 0,0 \rangle$, then $x=y=0$, which contradicts the constraint $x^2+y^2 = 1$.
  - If there is λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$, we have
  $$2xy = 2\lambda x, \quad x^2 = 2\lambda y$$
  so $2xy^2 = 2\lambda xy = x^3$. Thus either $x=0$ or $2y^2 = x^2$.
    * If $x=0$, then the value of f is 0.
    * If $2y^2 = x^2$, then the constraint becomes $3y^2 = 1$, or $y^2 = \frac{1}{3}$. So $y = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$, but the latter is excluded as we also have $y \geq 0$. So $y = \frac{1}{\sqrt{3}}$, $x^2 = \frac{2}{3}$, so the value of f is $\frac{2}{3\sqrt{3}}$.
- Step 2: Find the max/min values of f at the critical points. We did this earlier with Step 1.
- Step 3: Find the global max/min of f on the boundary. The boundary are when $y=0$, so $x=1$ or $x=-1$. In any case, the value of f is 0.
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. So the global maximum is $\frac{2}{3\sqrt{3}}$ and the global minimum is 0.

(2) Use the 4-step process.
- Step 1: Find the Lagrange critical points. Note $x^2+y^2=4$, so $f(x,y) = e^{-4}(4+y^2)$. So $\nabla f(x,y) = \langle 0, 2e^{-4}y \rangle$. Lagrange critical points are when $\nabla f(x,y) = \langle 0, 2e^{-4}y \rangle$ is parallel to $\nabla g(x,y)$ where $g(x,y) = x^2+y^2$. So $\nabla g(x,y) = \langle 2x, 2y \rangle$. This can only happen when $x=0$. So $y^2=4$, so $y=2$ or $y=-2$. Since $x+y \geq 0$, $y=2$. So $(0,2)$ is a Lagrange critical point.
- Step 2: Find the max/min values of f at the critical points. At $(0,2)$, the value of f is $8e^{-4}$.
- Step 3: Find the global max/min of f on the boundary. The boundary points are when $x+y=0$ and $x^2+y^2=4$. Since $y=-x$, so $2x^2=4$ or $x^2=2$ or $x=\sqrt{2}$ or $-\sqrt{2}$. Then $y^2=2$, so the value of f is $6e^{-4}$.
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. The global maximum is $8e^{-4}$ and the global minimum is $6e^{-4}$.

(3) Use the 4-step process.
- Step 1: Find the Lagrange critical points. They happen when $\nabla f(x,y,z) = \langle yz, xz, xy \rangle$ is parallel to $\nabla g(x,y,z)$ where $g(x,y,z) = x^2+y^2+z^2$. So $\nabla g(x,y,z) = \langle 2x, 2y, 2z \rangle$. This is the case when either $\nabla g(x,y,z)$ is zero, or there is a number λ such that $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$.
  - If $\nabla g(x,y,z)$ is zero, then $x=y=z=0$, which contradicts the constraint $x^2+y^2+z^2=3$.

– If there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, we have

$$yz = 2\lambda x, \quad xz = 2\lambda y, \quad xy = 2\lambda z$$

so

$$xyz = 2\lambda x^2, \quad xyz = 2\lambda y^2, \quad xyz = 2\lambda z^2$$

so $2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$. So either $\lambda = 0$ or $x^2 = y^2 = z^2$.

* If $\lambda = 0$, then $yz = xz = xy = 0$, so in any case $xyz = 0$.
* If $x^2 = y^2 = z^2$, then the constraint becomes $3x^2 = 3$, or $x^2 = 1$, or $x = 1$ or -1. So $y = 1$ or $y = -1$, and $z = 1$ or $z = -1$. Note $z \geq 0$, so $z = 1$ only happens. In any case, the values of f are either 1 or -1.

- Step 2: Find the max/min values of f at the critical points. The max is 1 and the min is -1.
- Step 3: Find the global max/min of f on the boundary. On the boundary, it is $x^2+y^2 = 3$ and $z = 0$. On that, $xyz = 0$.
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. The global max is 1 and the global min is -1.

□

Exercise 4. Find the global maximum and minimum values of f on the given region.

(1) $f(x, y) = x^3 - 12x + y^3 - 12y$ on the region

$$D = \{(x, y) \mid (x + 2)^2 + (y + 2)^2 \leq 13,\ x \geq -5\}$$

(2) $f(x, y) = x + y$ on the region

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ ex^2 \leq y \leq e^x\}$$

(3) $f(x, y, z) = x^4 + y + z^2$ on the region

$$D = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1,\ x \geq 0,\ y \geq 0\}$$

(4) $f(x, y, z) = xz + yz - xy$ on the region

$$D = \{(x, y, z) \mid z^2 \geq x^2 + y^2,\ (2 - z)^2 \geq x^2 + y^2,\ 0 \leq z \leq 2\}$$

Solution.

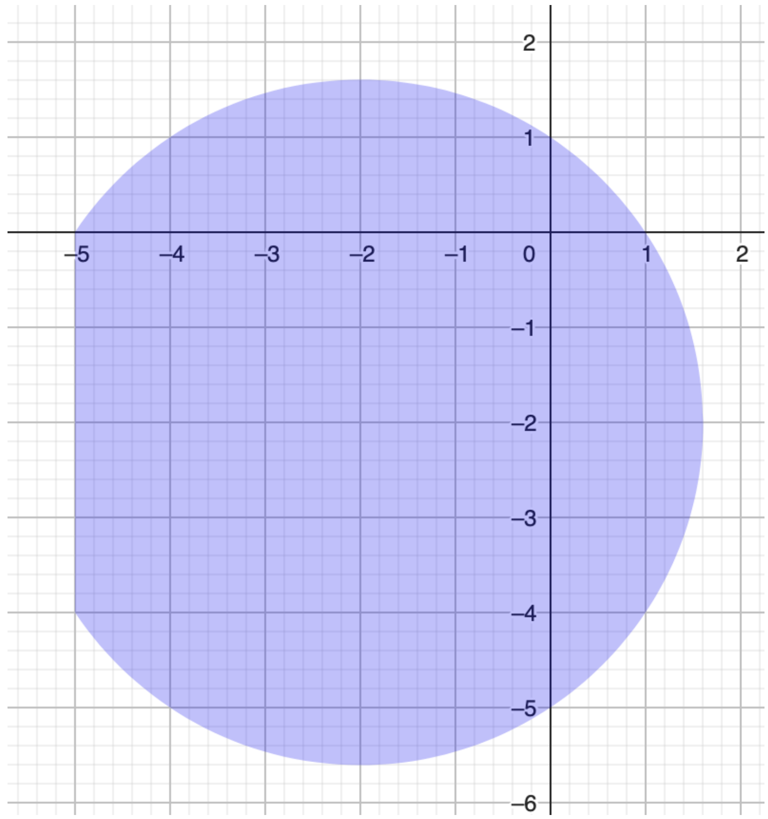
(1) Use the 4-step process.

- Step 1: Find the critical points. The critical points are when $\nabla f(x, y) = \langle 0, 0\rangle$, so

$$\langle 3x^2 - 12, 3y^2 - 12\rangle = \langle 0, 0\rangle$$

This happens when $x^2 = 4$ and $y^2 = 4$, so $x = 2$ or -2 and $y = 2$ and $y = -2$. Since $(x + 2)^2 + (y + 2)^2 \leq 13$, $(2, 2)$ is excluded, whereas $(2, -2)$, $(-2, 2)$ and $(-2, -2)$ are okay. So, there are three critical points, $(2, -2)$, $(-2, 2)$, $(-2, -2)$.

- Step 2: Find the max/min values of f at the critical points. At $(2, -2)$, $f(2, -2) = 8 - 24 - 8 + 24 = 0$. At $(-2, 2)$, $f(-2, 2) = -8 + 24 + 8 - 24 = 0$. At $(-2, -2)$, $f(-2, -2) = -8 + 24 - 8 + 24 = 32$. So, the max at the critical points is 32, and the min at the critical points is 0.
- Step 3: Find the global max/min of f on the boundary. The boundary is consisted of two parts, the vertical line $x = -5$ and the arc $(x + 2)^2 + (y + 2)^2 = 13$.



– If $x = -5$, then $(x+2)^2+(y+2)^2 \le 13$ implies $(y+2)^2 \le 4$, so $-4 \le y \le 0$. On this, the constrained optimization problem is to optimize $f(-5, y) = -125 + 60 + y^3 - 12y = y^3 - 12y - 65$ on $-4 \le y \le 0$.
  * Critical points are when $f'(y) = 3y^2 - 12 = 0$, or $y^2 = 4$, or $y = 2$ or -2. Since $-4 \le y \le 0$, $y = -2$ is the only possibility. At $y = -2$, $f(-2) = -8 + 24 - 65 = -49$.
  * Boundary points are $y = -4$ and $y = 0$, at which $f(-4) = -64+48-65 = -81$, and $f(0) = -65$.

  So, on the vertical line, the max is -49 and the min is -81.
– If $(x+2)^2+(y+2)^2 = 13$, the boundary points are $x = -5$, so $y = -4$ or $y = 0$. So $(-5, -4)$ and $(-5, 0)$ are boundary points. This is a constrained optimization with boundary, where the constraint is $g(x, y) = 13$, where $g(x, y) = (x+2)^2 + (y+2)^2$. So by Lagrange multipliers, we want $\nabla f(x, y) = \langle 3x^2 - 12, 3y^2 - 12\rangle$ parallel to $\nabla g(x, y) = \langle 2(x + 2), 2(y + 2)\rangle$. This happens either when ∇g is zero or there is λ such that $\nabla f(x, y) = \lambda \nabla g(x, y)$.
  * If $\nabla g(x, y)$ is zero, this means $x = -2$ and $y = -2$. This does not satisfy the constraint $(x+2)^2 + (y+2)^2 = 13$, so this possibility does not happen.
  * If there is λ such that $\nabla f(x, y) = \lambda \nabla g(x, y)$, we have

$$3(x^2 - 4) = 2\lambda(x + 2), \quad 3(y^2 - 4) = 2\lambda(y + 2).$$

    From the first equation, we have either $x + 2 = 0$ or $3(x - 2) = 2\lambda$. From the second equation, we have either $y + 2 = 0$ or $3(y - 2) = 2\lambda$. Thus either $x = -2$, $y = -2$ or $x = y$.

If $x=-2$, then the constraint says that $(y+2)^2=13$, so $y=-2+\sqrt{13}$ or $y=-2-\sqrt{13}$. We have $f(-2,-2+\sqrt{13})=-8+24+(-2+\sqrt{13})^3-12(-2+\sqrt{13})=-46+13\sqrt{13}$ and $f(-2,-2-\sqrt{13})=-46-13\sqrt{13}$.
If $y=-2$, then the constraint says that $(x+2)^2=13$, so $x=-2+\sqrt{13}$ or $x=-2-\sqrt{13}$. But $x=-2-\sqrt{13}$ is outside the range $x\geq -5$, so only $x=-2+\sqrt{13}$ is possible. Then $f(-2+\sqrt{13},-2)=-46+13\sqrt{13}$.
If $x=y$, then $(x+2)^2=\frac{13}{2}$, so $x=-2+\sqrt{\frac{13}{2}}$ or $-2-\sqrt{\frac{13}{2}}$. Both lie in $x\geq -5$, so the final Lagrange critical points are $(-2+\sqrt{\frac{13}{2}},-2+\sqrt{\frac{13}{2}})$ and $(-2-\sqrt{\frac{13}{2}},-2-\sqrt{\frac{13}{2}})$. Then

$$f(-2+\sqrt{\frac{13}{2}},-2+\sqrt{\frac{13}{2}})=2(-2+\sqrt{\frac{13}{2}})^3-24(-2+\sqrt{\frac{13}{2}})$$

$$=2(-8+12\sqrt{\frac{13}{2}}-6\cdot\frac{13}{2}+\frac{13}{2}\sqrt{\frac{13}{2}})+48-24\sqrt{\frac{13}{2}}$$

$$=2(-47+\frac{37}{2}\sqrt{\frac{13}{2}})+48-24\sqrt{\frac{13}{2}}$$

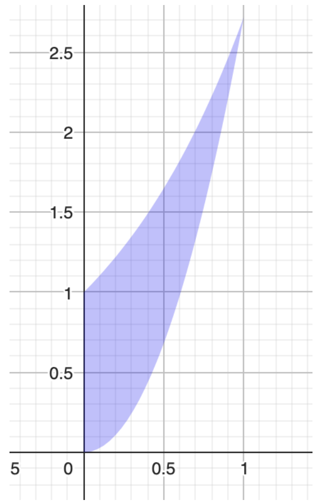
$$=-46+13\sqrt{\frac{13}{2}}$$

and $f(-2-\sqrt{\frac{13}{2}},-2-\sqrt{\frac{13}{2}})=-46-13\sqrt{\frac{13}{2}}$.

Thus, the max on the arc is $-46+13\sqrt{13}$ and the min on the arc is $-46-13\sqrt{13}$. So the max on the boundary is $-46+13\sqrt{13}$ and the min on the boundary is $-46-13\sqrt{13}$.

- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. The global max is 32 and the global min is $-46-13\sqrt{13}$.

(2) Use the 4-step process.
- Step 1: Find the critical points. As $\nabla f(x,y)=\langle 1,1\rangle$, there is no critical point.
- Step 2 is thus skipped.
- Step 3: Find the global max/min of f on the boundary. The boundary is divided into three parts, the vertical line $\{(0,y)\mid 0\leq y\leq 1\}$, the upper curve $\{(x,y)\mid 0\leq x\leq 1, y=e^x\}$, and the lower curve $\{(x,y)\mid 0\leq x\leq 1, y=ex^2\}$.



- On the vertical line, the function becomes $f(y) = y$, and its max on $0 \leq y \leq 1$ is obviously 1 and its min is 0.
- On the upper curve $\{(x, y) \mid 0 \leq x \leq 1, y = e^x\}$, we are solving constrained optimization of $f(x, y)$ with constraint $g(x, y) = 0$ where $g(x, y) = y - e^x$, with boundary. Note the boundary points are when $x = 0$ and $x = 1$, which corresponds to $y = 1$ and $y = e$. The Lagrange critical points happen when $\nabla f(x, y) = \langle 1, 1 \rangle$ is parallel to $\nabla g(x, y) = \langle -e^x, 1 \rangle$. This happens only if $-e^x = 1$, which is impossible, so there is no Lagrange critical point. On the boundary, $(0, 1)$ gives $f(0, 1) = 1$, and on $(1, e)$, $f(1, e) = e + 1$. So the max is $e + 1$ and the min is 1.
- On the lower curve $\{(x, y) \mid 0 \leq x \leq 1, y = ex^2\}$, we are solving constrained optimization of $f(x, y)$ with constraint $g(x, y) = 0$ where $g(x, y) = y - ex^2$, with boundary. Note the boundary points are when $x = 0$ and $x = 1$, which corresponds to $y = 0$ and $y = e$. The Lagrange critical points happen when $\nabla f(x, y) = \langle 1, 1 \rangle$ is parallel to $\nabla g(x, y) = \langle -2ex, 1 \rangle$. This happens only if $-2ex = 1$, or $x = -\frac{1}{2e}$, which is out of range, so there is no Lagrange critical point. On the boundary, $(0, 0)$ gives $f(0, 0) = 0$, and on $(1, e)$, $f(1, e) = e + 1$, so the max is $e + 1$ and the min is 0.

So the max on the boundary is $e + 1$ and the min on the boundary is 0.

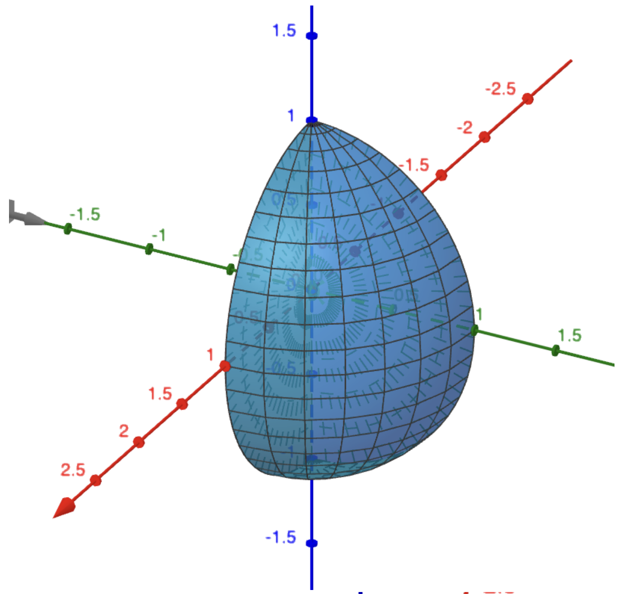
- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. So the global max is $e + 1$ and the global min is 0.

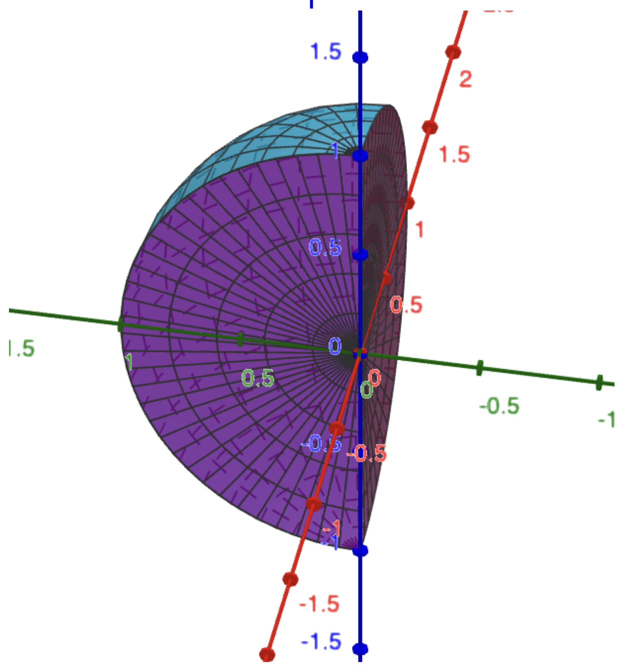
(3) Use the 4-step process.

- Step 1: Find the critical points. Note that

$$\nabla f(x, y, z) = \langle 4x^3, 1, 2z \rangle,$$

so this is never a zero vector as the second component is 1. So there is no critical point.
- Step 2 is skipped.
- Step 3: Find the global max/min of f on the boundary. The boundary is divided into three parts, the spherical part $x^2 + y^2 + z^2 = 1$, $x, y \geq 0$, the xz-plane part, namely $y = 0$, $x^2 + z^2 \leq 1$ and $x \geq 0$, and the yz-plane part, namely $x = 0$, $y^2 + z^2 \leq 1$ and $y \geq 0$.





- The problem over the spherical part $x^2+y^2+z^2=1, x, y \geq 0$ is a constrained optimization with boundary. The constraint function is $g(x,y,z)=x^2+y^2+z^2$, which is set to be 1. Thus the Lagrange critical points are obtained when $\nabla f(x,y,z)=\langle 4x^3, 1, 2z\rangle$ is parallel to $\nabla g(x,y,z)=\langle 2x, 2y, 2z\rangle$. This happens either when ∇g is zero or there is λ such that $\langle 4x^3, 1, 2z\rangle = \lambda\langle 2x, 2y, 2z\rangle$. The former case happens when $x=y=z=0$, which is not allowed because of the constraint $x^2+y^2+z^2=1$, so the latter case is the only possibility, where we have
$$4x^3 = 2\lambda x, 1 = 2\lambda y, 2z = 2\lambda z.$$
The third equation tells either $\lambda = 1$ or $z=0$. If $\lambda=1$, then we have
$$4x^3 = 2x, \quad 1 = 2y,$$

so $y = \frac{1}{2}$, and either $x = 0$ or $2x^2 = 1$, so $x^2 = \frac{1}{2}$, so $x = \frac{1}{\sqrt{2}}$ (because $x \geq 0$). The case $x = 0$ or $z = 0$ is on the boundary so they will be dealt later anyways. Thus we only need to consider $y = \frac{1}{2}$ and $x = \frac{1}{\sqrt{2}}$. Then $x^2 + y^2 + z^2 = 1$ gives $z^2 = \frac{1}{4}$, so $z = \frac{1}{2}$ or $z = -\frac{1}{2}$. Thus we get points $(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{2}, -\frac{1}{2})$, on which we have $f = 1$.
The boundary is naturally divided into two parts, $x = 0$, $y = 0$.

* If $x = 0$, the function to be maximized/minimized is $f(y, z) = y + z^2$, with constraint $y^2 + z^2 = 1$, $y \geq 0$. This again is constrained optimization. Note Lagrange critical points happen when $\nabla f(y, z) = \langle 1, 2z \rangle$ is parallel to $\nabla g(y, z) = \langle 2y, 2z \rangle$, where $g(y, z) = y^2 + z^2$. So either ∇g is zero or there is λ such that $\nabla f(y, z) = \lambda \nabla g(y, z)$.
  If $\nabla g(y, z) = \langle 0, 0 \rangle$, then $y = z = 0$, which is not allowed because of the constraint $y^2 + z^2 = 1$. So there is λ such that $\nabla f(y, z) = \lambda \nabla g(y, z)$, which means
  $$1 = 2\lambda y, \quad 2z = 2\lambda z$$
  From the second equation, either $z = 0$ or $\lambda = 1$. If $z = 0$, then $y^2 = 1$, so $y = 1$ (by $y \geq 0$), so the value of f is 1. If $\lambda = 1$, then $y = \frac{1}{2}$, so $z^2 = \frac{3}{4}$, so $z = \frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$. So the value of f is $\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$. There is boundary, $y = 0$. If $y = 0$, then $z^2 = 1$, so $z = 1$ or $z = -1$, so the value of f is 1. So the max on this boundary is $\frac{5}{4}$, and the min on this boundary is 1.
* If $y = 0$, the function to be maximized/minimized is $f(x, z) = x^4 + z^2$, with constraint $x^2 + z^2 = 1$, $x \geq 0$. This is constrained optimization. Lagrange critical points happen when $\nabla f(x, z) = \langle 4x^3, 2z \rangle$ is parallel to $\nabla g(x, z) = \langle 2x, 2z \rangle$, where $g(x, z) = x^2 + z^2$. So either ∇g is zero or there is λ such that $\nabla f(x, z) = \lambda \nabla g(x, z)$.
  If $\nabla g(x, z) = \langle 0, 0 \rangle$, then $x = z = 0$, which is not allowed because of the constraint $x^2 + z^2 = 1$. So there is λ such that $\nabla f(x, z) = \lambda \nabla g(x, z)$, which means
  $$4x^3 = 2\lambda x, \quad 2z = 2\lambda z$$
  The second equation tells either $z = 0$ or $\lambda = 1$. If $z = 0$, then $y = z = 0$ implies $x^2 = 1$ or $x = 1$ (because $x \geq 0$). Then the value of f is 1. If $\lambda = 1$, then $4x^3 = 2x$, so either $x = 0$ or $2x^2 = 1$, or $x = \frac{1}{\sqrt{2}}$ (by $x \geq 0$). The value at $x = 0$ is 1, and the value at $x = \frac{1}{\sqrt{2}}$ is, as $y = 0$, $z = \frac{1}{\sqrt{2}}$, so the value is $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. We've already included the boundary which are $x = 0$. So the max is 1 and the min is $\frac{3}{4}$.

So the max on the boundary of spherical part is $\frac{5}{4}$ and the min on the boundary of spherical part is $\frac{3}{4}$. Since critical point has value 1, these are the global max and min on the spherical part.

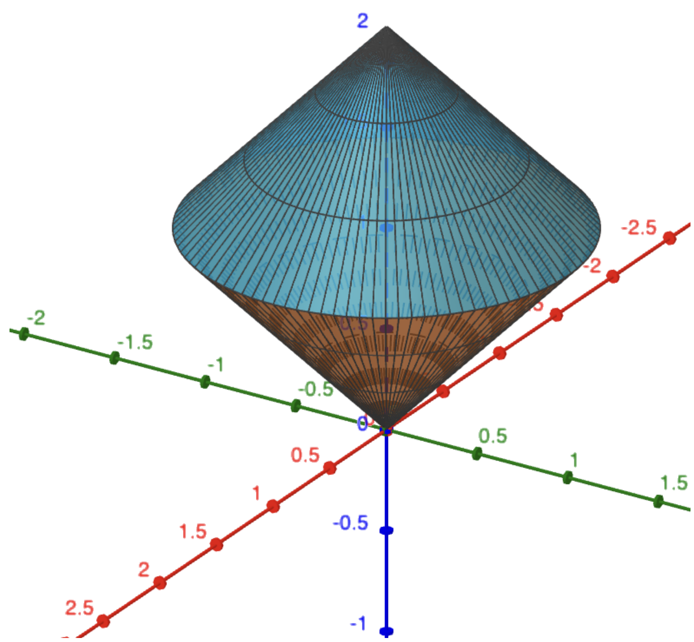
- If $y = 0$, then f becomes $f(x, z) = x^4 + z^2$, and the region becomes $x^2 + z^2 \leq 1$ with $x \geq 0$. The boundary that is not considered in the boundary of spherical part is $x = 0$, $y = 0$. Then $f(z) = z^2$ with $z^2 \leq 1$, so the max is 1 and the min is 0. For the critical points, we want $\langle 4x^3, 2z \rangle = \langle 0, 0 \rangle$, which happens at $(0, 0)$ anyways, so the new value is 0.

- If $x = 0$, then f becomes $f(y, z) = y + z^2$, and the region becomes $y^2 + z^2 \leq 1$ with $y \geq 0$. The boundaries are all already considered, and the critical point doesn't happen as $f_y(y, z) = 1 \neq 0$.

- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. So we conclude that the max is $\frac{5}{4}$ and the min is 0.

(4) Use the 4-step process.

- Step 1: Find the critical points. The critical points happen when ∇f is zero, so $\langle z - y, z - x, x + y\rangle = \langle 0, 0, 0\rangle$. This happens when $x = y = z = 0$. This is indeed in the region, and the value is 0.
- Step 2: Find the max/min values of f at the critical points. We did this above.
- Step 3: Find the global max/min of f on the boundary. The boundary is naturally divided into two parts, $z^2 = x^2 + y^2$ and $0 \leq z \leq 1$, and $(2 - z)^2 = x^2 + y^2$ and $1 \leq z \leq 2$.



- In the first region, the Lagrange critical points happen when $\langle z-y, z-x, x+y\rangle$ is parallel to $\langle 2x, 2y, -2z\rangle$. If $\langle 2x, 2y, -2z\rangle$ is zero, then $x = y = z = 0$ and this is something we already considered. If not,

$$\langle z - y, z - x, x + y\rangle = \lambda\langle 2x, 2y, -2z\rangle$$

or

$$z - y = 2\lambda x, \quad z - x = 2\lambda y, \quad x + y = -2\lambda z.$$

Adding the first two we get

$$2z - x - y = 2\lambda(x + y)$$

or

$$2z = (2\lambda + 1)(x + y) = -2\lambda(2\lambda + 1)z$$

so $(4\lambda^2+2\lambda+2)z = 0$. Since $4\lambda^2+2\lambda+2 > 0$, $z = 0$. Then $z^2 = x^2+y^2$ implies $x = y = 0$, which is already considered.
The boundary are $x^2+y^2 = 1$ with $z = 1$, and $(0,0,0)$. The latter is already considered, and the former gives you a function $f(x,y) = x+y-xy$ to be optimized on $x^2+y^2 = 1$. This is again Lagrange multipliers, where we want

$$\langle 1-y, 1-x\rangle = \lambda\langle 2x, 2y\rangle$$

or

$$1-y = 2\lambda x, \quad 1-x = 2\lambda y$$

Subtracting, we get

$$x-y = 2\lambda(x-y)$$

so either $x = y$ or $\lambda = \frac{1}{2}$. If $\lambda = \frac{1}{2}$ we have $1 = x+y$. So either $x = y$ or $x+y = 1$. If $x = y$ we have $x = y = \pm\frac{1}{\sqrt{2}}$ from the constraints, from which we have $f = \pm\sqrt{2}-2$. If $x+y = 1$, from constraints we have

$$x^2+(1-x)^2 = 1,$$

or $-2x+2x^2 = 0$, so either $x = 0$ or $x = 1$. At $(0,1)$ we have $f = 1$ and at $(1,0)$ again $f = 1$. So on this boundary the max is 1 and the min is $-\sqrt{2}-2$.

  – In the second region, the Lagrange critical points happen when $\langle z-y, z-x, x+y\rangle$ is parallel to $\langle 2x, 2y, -2z+4\rangle$. If $\langle 2x, 2y, 4-2z\rangle$ is zero, then $x = y = 0$ and $z = 2$, which gives $f = 0$. If not,

$$\langle z-y, z-x, x+y\rangle = \lambda\langle 2x, 2y, 4-2z\rangle$$

$$z-y = 2\lambda x, \quad z-x = 2\lambda y, \quad x+y = 4\lambda - 2\lambda z.$$

Adding the first two we get

$$2z-x-y = 2\lambda(x+y)$$

or

$$2z = (2\lambda+1)(x+y) = -2\lambda(2\lambda+1)z$$

so $(4\lambda^2+2\lambda+2)z = 0$. Since $4\lambda^2+2\lambda+2 > 0$, $z = 0$. This is not in the region. The boundary are either the circle we already considered or $(0,0,2)$, which is also considered.

- Step 4: Compare the values from Step 2 and Step 3 and take the max/min. Thus the max is 1 and the min is $-\sqrt{2}-2$.

□

19. Lagrange multipliers II: Multiple constraints

Exercise 1. Find the boundary of the region, and divide it naturally into parts.

(1) $\{(x,y) \mid 0 \le x+y \le 1\}$
(2) $\{(x,y) \mid x^2+4y^2 \le 4,\ x \ge 1\}$
(3) $\{(x,y) \mid x+2y^2 \le 0,\ x+y \le -1\}$
(4) $\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 2\}$
(5) $\{(x,y,z) \mid x^2+y^2+z^2 \le 1,\ x+y \le 1,\ x \ge \frac{1}{2}\}$
(6) $\{(x,y,z) \mid x^2+y^2 = z^2,\ x+y \ge 1,\ z \le 5\}$

Solution.

(1) It divides into two parts,

$$\{(x,y) \mid x+y=0\}$$

$$\{(x,y) \mid x+y=1\}$$

(2) It divides into two parts,

$$\{(x,y) \mid x^2+4y^2=4,\ x\geq 1\}$$

$$\{(x,y) \mid x^2+4y^2\leq 4,\ x=1\}$$

(3) It divides into two parts,

$$\{(x,y) \mid x+2y^2=0,\ x+y\leq -1\}$$

$$\{(x,y) \mid x+2y^2\leq 0,\ x+y=-1\}$$

(4) It divides into four parts,

$$\{(x,y) \mid x=0,\ 0\leq y\leq 2\}$$

$$\{(x,y) \mid x=2,\ 0\leq y\leq 2\}$$

$$\{(x,y) \mid 0\leq x\leq 2,\ y=0\}$$

$$\{(x,y) \mid 0\leq x\leq 2,\ y=2\}$$

(5) It divides into three parts,

$$\{(x,y,z) \mid x^2+y^2+z^2=1,\ x+y\leq 1,\ x\geq \frac{1}{2}\}$$

$$\{(x,y,z) \mid x^2+y^2+z^2\leq 1,\ x+y=1,\ x\geq \frac{1}{2}\}$$

$$\{(x,y,z) \mid x^2+y^2+z^2\leq 1,\ x+y\leq 1,\ x=\frac{1}{2}\}$$

(6) It divides into two parts,

$$\{(x,y,z) \mid x^2+y^2=z^2,\ x+y=1,\ z\leq 5\}$$

$$\{(x,y,z) \mid x^2+y^2=z^2,\ x+y\geq 1,\ z=5\}$$

□

Exercise 2. Determine whether there is a global maximum or a global minimum of a function f on a region D, and if they exist, find the values.

(1) $f(x,y,z)=z$ on $D=\{(x,y,z) \mid x^2+y^2+z^2=1,\ x+y-z=0\}$
(2) $f(x,y,z)=x^2+y^2$ on $D=\{(x,y,z) \mid x^2+y^2+z^2=50,\ x-z=0\}$

Solution.

(1) Note that the region is compact. The constraints are $g(x,y,z) = 1$ and $h(x,y,z) = 0$ where $g(x,y,z) = x^2+y^2+z^2$ and $h(x,y,z) = x+y-z$. Thus

$$\nabla f(x,y,z) = \langle 0,0,1\rangle, \quad \nabla g(x,y,z) = \langle 2x,2y,2z\rangle, \quad \nabla h(x,y,z) = \langle 1,1,-1\rangle$$

Lagrange critical points are when either $\nabla g(x,y,z)$ or $\nabla h(x,y,z)$ are zero, or $\nabla f(x,y,z) = \lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$. Note that $\nabla h(x,y,z)$ is not zero, and $\nabla g(x,y,z) = \langle 2x,2y,2z\rangle$ is zero if $x=y=z=0$, which does not lie in D because of the constraint $x^2+y^2+z^2 = 1$. Thus we need to solve

$$\nabla f(x,y,z) = \lambda\nabla g(x,y,z) + \mu\nabla h(x,y,z)$$

or

$$0 = 2\lambda x + \mu, \quad 0 = 2\lambda y + \mu, \quad 1 = 2\lambda z - \mu$$

From the first two equations, $2\lambda x = 2\lambda y$, so either $\lambda = 0$ or $x = y$.

- If $\lambda = 0$, we have $\mu = 0$ and $1 = -\mu$, which is a contradiction.
- If $x = y$, the constraints say $2x^2+z^2 = 1$ and $2x = z$. Thus, $6x^2 = 1$, or $x = \frac{1}{\sqrt{6}}$ or $-\frac{1}{\sqrt{6}}$. Thus the Lagrange critical points are $(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$ and $(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$.

As there is no boundary, the global maximum is $\frac{2}{\sqrt{6}}$, and the global minimum is $-\frac{2}{\sqrt{6}}$.

(2) Since the region is compact, we use the 4-step process.

- Step 1. Find the Lagrange critical points. The constraints are $g(x,y,z) = 50$ and $h(x,y,z) = 0$ where $g(x,y,z) = x^2+y^2+z^2$ and $h(x,y,z) = x-z$. This happens either when $\nabla g(x,y,z) = \langle 2x,2y,2z\rangle$ or $\nabla h(x,y,z) = \langle 1,0,-1\rangle$ is zero, or when $\nabla f(x,y,z) = \lambda\nabla g(x,y,z) + \mu\nabla h(x,y,z)$. The former case can only happen when $2x = 2y = 2z = 0$, which does not satisfy $x^2+y^2+z^2 = 50$. Thus we need to solve

$$\nabla f(x,y,z) = \lambda\nabla g(x,y,z) + \mu\nabla h(x,y,z)$$

or

$$\langle 2x,2y,0\rangle = \lambda\langle 2x,2y,2z\rangle + \mu\langle 1,0,-1\rangle$$

or

$$2x = 2\lambda x + \mu, \quad 2y = 2\lambda y, \quad 0 = 2\lambda z - \mu$$

From the second equation, either $\lambda = 1$ or $y = 0$.

  - If $\lambda = 1$, we have $2x = 2x+\mu$ or $0 = 2z-\mu$. So, $\mu = 0$ and $z = 0$. Then $x - z = 0$ means $x = 0$, so $y^2 = 50$. Thus the Lagrange critical points are $(0,\sqrt{50},0)$ and $(0,-\sqrt{50},0)$.
  - If $y = 0$, then $x^2+z^2 = 2x^2 = 50$, so the Lagrange critical points are $(5,0,5)$ and $(-5,0,-5)$.
- Step 2. Evaluate on the Lagrange critical points. We have $f(0,\sqrt{50},0) = 50$, $f(0,-\sqrt{50},0) = 50$, $f(5,0,5) = 25$, $f(-5,0,-5) = 25$.
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Compare. The global max is 50 and the global min is 25.

$\square$

Exercise 1. Determine whether the following region is closed, bounded or compact.

(1) $\{(x, y) \mid x^2 + y^2 \leq 1\}$
(2) $\{(x, y) \mid x^2 + y^2 < 1\}$
(3) $\{(x, y) \mid x + y = 0\}$
(4) $\{(x, y) \mid x^3 + y^3 \leq 1\}$
(5) $\{(x, y) \mid x^4 + y^2 \leq 1\}$
(6) $\{(x, y) \mid x^2 + y^4 + x \leq 1,\ y \geq 0\}$
(7) $\{(x, y, z) \mid x^2 + y^2 + z \leq 1\}$
(8) $\{(x, y, z) \mid x^2 + y^4 \leq z^2\}$
(9) $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 2x + 2y + 2z,\ z \geq 0\}$

Solution.

(1) Closed and bounded, so compact.
(2) Not closed but bounded, so not compact.
(3) Closed but not bounded (x can go to ∞), so not compact.
(4) Closed but not bounded (x can go to $+\infty$ while y goes to $-\infty$), so not compact.
(5) Closed and bounded ($|x| \leq 1$ and $|y| \leq 1$), so compact.
(6) Closed. To see if it is bounded, note $x^2 + y^4 + x \leq 1$ is equivalent to $(x^2 + x + \frac{1}{4}) + y^4 \leq \frac{5}{4}$, or $(x + \frac{1}{2})^2 + y^4 \leq \frac{5}{4}$, so it is bounded. So it is compact.
(7) Closed, but not bounded (z can go to $-\infty$), so not compact.
(8) Closed, but not bounder (all x, y, z can go to ∞), so not compact.
(9) Closed. To see if it is bounded, note $x^2 + y^2 + z^2 \leq 2x + 2y + 2z$ can be written as
$$(x^2 - 2x) + (y^2 - 2y) + (z^2 - 2z) \leq 0,$$
or
$$(x^2 - 2x + 1) + (y^2 - 2y + 1) + (z^2 - 2z + 1) \leq 3,$$
or $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 \leq 3$, so it is bounded. So it is compact.

□

Exercise 2. Determine whether f has a global maximum and/or minimum on the region, and if they exist, find the values.

(1) $f(x, y) = xy + x + y$, on $y \geq x^2$
(2) $f(x, y) = x^2 + y^2$, on $xy \geq 1$
(3) $f(x, y) = x^2 + 3y^2 - 4x - 6y$, on $x \geq 0$, $y \geq 0$
(4) $f(x, y) = xye^{-x^2-y^2}$, on $2x - y = 0$
(5) $f(x, y) = x^3 + y^3 - 3xy$, on all real numbers x, y
(6) $f(x, y) = 2x^2 - 2xy + y^2 - 2x$, on all real numbers x, y
(7) $f(x, y) = x^2 + 2y$, on $2x + y^2 \leq 3$
(8) $f(x, y) = e^{xy}$, on $x^3 + y^3 = 16$
(9) $f(x, y, z) = xyz$, on $xy + 2yz + 2zx = 12$, $x, y, z \geq 0$
(10) $f(x, y, z) = 4x + 2y + z$, on $x^2 + y + z^2 = 1$
(11) $f(x, y, z) = x\ln(x) + y\ln(y) + z\ln(z) - \frac{x+y+z}{3}\ln(xyz)$, on $x, y, z \geq 1$

Solution. Recall that the "extended 4-step process" is as follows.

- Find critical points.

- Evaluate on critical points.
- Find the max/min on the boundary.
- Compare to find the max/min candidates.
- Identify the "boundary at infinity."
- Find the max/min on the "boundary at infinity."
- Compare to determine global max/min.

(1) This is (unconstrained) optimization. We use extended 4-step process.
  - Find critical points. Since $\nabla f(x,y) = \langle y+1, x+1\rangle$, the critical point happens when $y=-1$ and $x=-1$. Since this lies in the region, $(-1,-1)$ is indeed a critical point.
  - Evaluate on critical points. At $(-1,-1)$, $f(-1,-1)=-1$.
  - Find the max/min on the boundary. The boundary is given by $y = x^2$. This is a constrained optimization problem, where the region is again closed but not bounded. So we use extended 4-step process.
    - Find critical points. Here this means we need to find Lagrange critical points, where the constraint is $g(x,y)=0$, $g(x,y)=y-x^2$. So we want $\nabla f(x,y) = \langle y+1, x+1\rangle$ to be parallel to $\nabla g(x,y) = \langle -2x, 1\rangle$. This happens either when $\nabla g(x,y) = \langle 0,0\rangle$ or there is λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$.
      * Since the second component of $\nabla g(x,y)$ is 1, this can be never zero.
      * If $\nabla f(x,y) = \lambda\nabla g(x,y)$ holds, then $y+1 = -2\lambda x$ and $x+1=\lambda$. So, $x = \lambda - 1$, and $y = -2\lambda x - 1 = -2\lambda(\lambda-1)-1 = -2\lambda^2+2\lambda-1$. We are under the constraint that $y = x^2$, so $-2\lambda^2+2\lambda-1 = (\lambda-1)^2 = \lambda^2-2\lambda+1$, so $3\lambda^2-4\lambda+2=0$. But this is impossible, since the discriminant is $16-24<0$.

      Thus there are no (Lagrange) critical points.
    - Evaluate on critical points. This is skipped because there are no critical points.
    - Find the max/min on the boundary. There is no boundary, so this is skipped.
    - Compare to find the max/min candidates. There are no max/min candidates. Already you see here that there are no global max/min on the boundary. Thus, the rest of the extended 4-step process is unnecessary.
  - Compare to find the max/min candidates. The max candidate is -1, and the min candidate is also -1.
  - Identify the "boundary at infinity." The boundary at infinity is such that $y=+\infty$ (if x goes to $-\infty$ or $+\infty$, then y has to be automatically $+\infty$ by $y \geq x^2$, so this case contains all the boundary at infinity).
  - Find the max/min on the "boundary at infinity." If both x and y go to $+\infty$, which is certainly possible, $f(x,y)$ goes to $+\infty$ as all the terms xy, x, y go to $+\infty$. So the max on the boundary at infinity is $+\infty$. If $x=-2$ while y goes to $+\infty$, then $f(x,y) = -2y-2+y = -y-2$ goes to $-\infty$. So, the min on the boundary at infinity is $-\infty$.
  - Compare to determine global max/min. Because of the previous step, nothing can be global max or min. So there are no global max/min.

(2) We use extended 4-step process.
  - Find critical points. Note $\nabla f(x,y) = \langle 2x, 2y\rangle$, so the critical points are when $x = y = 0$. But this doesn't appear in the region $xy \geq 1$, so there are no critical points.
  - Evaluate on critical points. This is skipped because there are no critical points.

- Find the max/min on the boundary. The boundary is $xy = 1$, which is again closed but not bounded region. So this is a new constrained optimization problem.
  - Find critical points. This means we need to find Lagrange critical points, with constraint $g(x, y) = 1$, $g(x, y) = xy$. Since $\nabla g(x, y) = \langle y, x\rangle$, for ∇f and ∇g to be parallel, either ∇g is zero or there is λ such that $\nabla f = \lambda \nabla g$.
    * If $\nabla g(x, y) = \langle 0, 0\rangle$, this means $x = y = 0$. But this doesn't satisfy the constraint $xy = 1$, so this is not possible.
    * If $\nabla f = \lambda \nabla g$, then $2x = \lambda y$ and $2y = \lambda x$. Substituting, we get $2x = \lambda y = \frac{\lambda^2}{2}x$, so either $x = 0$ or $2 = \frac{\lambda^2}{2}$. Note however $x = 0$ is impossible under the constraint $xy = 1$. If $2 = \frac{\lambda^2}{2}$, $\lambda^2 = 4$, so either $\lambda = 2$ or $\lambda = -2$. If $\lambda = 2$, then $x = y$, so $xy = 1$ implies $x^2 = 1$, so either $x = y = 1$ or $x = y = -1$. If $\lambda = -2$ then $x = -y$. Then the constraint becomes $-x^2 = 1$, which is impossible.

    So the (Lagrange) critical points are $(1, 1)$ and $(-1, -1)$.
  - Evaluate on critical points. By above, $f(1, 1) = 2$, and $f(-1, -1) = 2$.
  - Find the max/min on the boundary. There is no boundary to $xy = 1$.
  - Compare to find the max/min candidates. The max candidate is 2 and the min candidate is 2.
  - Identify the "boundary at infinity." The boundary at infinity is either $x = \pm\infty$, at which y has to be 0, or $y = \pm\infty$, at which x has to be 0.
  - Find the max/min on the "boundary at infinity." Either way, the value of $f(x, y)$ on these boundary at infinity points is always $+\infty$, so the max on the boundary at infinity is $+\infty$, and the min on the boundary at infinity is $+\infty$.
  - Compare to determine global max/min. From above, we see there is no global maximum, while the global minimum is 2.
- Compare to find the max/min candidates. Thus the max candidate and min candidate are both 2.
- Identify the "boundary at infinity." The boundary at infinity can be anything involving ∞, as long as the signs of x, y match.
- Find the max/min on the "boundary at infinity." On the boundary at infinity, $f = +\infty$.
- Compare to determine global max/min. From above, there is no global maximum while 2 is the global minimum.

(3) We use extended 4-step process.
- Find critical points. Note $\nabla f(x, y) = \langle 2x - 4, 6y - 6\rangle$, so the critical point candidate is $(2, 1)$. This is in the region, so this is really a critical point.
- Evaluate on critical points. We have $f(2, 1) = -7$.
- Find the max/min on the boundary. The boundary is consisted of two parts, $x = 0$ and $y \geq 0$, and $y = 0$ and $x \geq 0$.
  - If $x = 0$ and $y \geq 0$, $f(x, y)$ becomes $f(y) = 3y^2 - 6y$. This is another unconstrained optimization problem, so we use the extended 4-step process.
    * Find critical points. Since $f'(y) = 6y - 6$, critical point is $y = 1$. This appears at $y \geq 0$, so this is a critical point.
    * Evaluate on critical points. We have $f(1) = -3$.
    * Find the max/min on the boundary. The boundary is $y = 0$, where the value is $f(0) = -6$.

    * Compare to find the max/min candidates. The max candidate is -3, and the min candidate is -6.
    * Identify the "boundary at infinity." The boundary at infinity is just $(0, +\infty)$, which gives $f = +\infty$ (the largest power of y only matters).
    * Find the max/min on the "boundary at infinity." We did this above.
    * Compare to determine global max/min. So the global max does not exist, and the global min is -6.
  - Compare to find the max/min candidates. The global max candidate is -6, while the global min candidate is -7.
  - Identify the "boundary at infinity." The boundary at infinity is either $x = +\infty$ or $y = +\infty$.
  - Find the max/min on the "boundary at infinity." In any case, $f = +\infty$.
  - Compare to determine global max/min. So the global maximum does not exist, and the global minimum is -7.

(4) We use extended 4-step process.
  - Find critical points. In this case we are looking for Lagrange critical points. Note that
$$\begin{aligned}\nabla f(x,y) &= \langle ye^{-x^2-y^2} - 2x^2ye^{-x^2-y^2}, xe^{-x^2-y^2} - 2xy^2e^{-x^2-y^2}\rangle \\ &= \langle y(1-2x^2)e^{-x^2-y^2}, x(1-2y^2)e^{-x^2-y^2}\rangle\end{aligned}$$
and
$$\nabla g(x,y) = \langle 2, -1\rangle$$
where $g(x,y) = 2x - y$ is set to be 0 as a constraint. Lagrange critical points are when $\nabla f(x,y)$ is parallel to $\nabla g(x,y)$. Since $\nabla g(x,y)$ is never zero, this happens when there is λ such that $\nabla f(x,y) = \lambda \nabla g(x,y)$. This means that
$$y(1-2x^2)e^{-x^2-y^2} = 2\lambda, \quad x(1-2y^2)e^{-x^2-y^2} = -\lambda$$
so
$$(y(1-2x^2) + 2x(1-2y^2))e^{-x^2-y^2} = 0$$
so
$$y(1-2x^2) + 2x(1-2y^2) = 0$$
Using the constraint $y = 2x$, this becomes
$$2x(1-2x^2) + 2x(1-8x^2) = 0$$
so
$$2x(2-10x^2) = 0$$
so either $x = 0$ or $2 - 10x^2 = 0$, so either $x = 0$, $\frac{1}{\sqrt{5}}$ or $-\frac{1}{\sqrt{5}}$. So Lagrange critcial points are $(0,0)$, $(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$ and $(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$.
  - Evaluate on critical points. On the Lagrange critical points, $f(0,0) = 0$, $f(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = \frac{2}{5}e^{-1}$ and $f(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}) = \frac{2}{5}e^{-1}$.
  - Find the max/min on the boundary. There is no boundary.
  - Compare to find the max/min candidates. The max candidate is $\frac{2}{5}e^{-1}$ and the min candidate is 0.
  - Identify the "boundary at infinity." The boundary at infinity are $(+\infty, +\infty)$ and $(-\infty, -\infty)$.

- Find the max/min on the "boundary at infinity." Note that $\lim_{x\to+\infty} xe^{-x^2} = \lim_{x\to+\infty} \frac{x}{e^{x^2}}$, which by L'Hopital is equal to $\lim_{x\to+\infty} \frac{1}{2xe^{x^2}} = 0$. Thus, the value at $(+\infty, +\infty)$ is 0. Similarly, $\lim_{x\to-\infty} xe^{-x^2} = 0$, so the value at $(-\infty, -\infty)$ is 0 as well. So, the max/min on the boundary at infinity are 0.
- Compare to determine global max/min. The max/min candidates are thus honest global max/min, so the global max is $\frac{2}{5e}$ and the global min is 0.

(5) We use extended 4-step process.
- Find critical points. Note

$$\nabla f(x, y) = \langle 3x^2 - 3y, 3y^2 - 3x\rangle$$

so the critical points happen when

$$3x^2 - 3y = 0, \quad 3y^2 - 3x,$$

or if $x^2 = y$ and $y^2 = x$. This means that $y = x^2 = y^4$. Thus, either $y = 0$ or $1 = y^3$. Thus, either $y = 0$ or $y = 1$. In turn, the critical points are $(0, 0)$ and $(1, 1)$.
- Evaluate on critical points. We have $f(0, 0) = 0$ and $f(1, 1) = -1$.
- Find the max/min on the boundary. There is no boundary.
- Compare to find the max/min candidates. The max candidate is 0 and the min candidate is -1.
- Identify the "boundary at infinity." The boundary at infinity is anything that has either x or y equal to $+\infty$ or $-\infty$.
- Find the max/min on the "boundary at infinity." At $(+\infty, 0)$, f is $+\infty$, and at $(-\infty, 0)$, f is $-\infty$, so the max on the boundary at infinity is $+\infty$, and the min on the boundary at infinity is $-\infty$.
- Compare to determine global max/min. Thus there are no global max or min.

(6) We use extended 4-step process.
- Find critical points. Note that

$$\nabla f(x, y) = \langle 4x - 2y - 2, -2x + 2y\rangle$$

so the critical points happen when $4x - 2y - 2 = 0$ and $-2x + 2y = 0$. So this means $y = x$, so $2x - 2 = 0$, or $x = y = 1$.
- Evaluate on critical points. At the critical point, $f(1, 1) = -1$.
- Find the max/min on the boundary. There is no boundary.
- Compare to find the max/min candidates. The max candidate and the min candidate are both -1.
- Identify the "boundary at infinity." The boundary at infinity is anything that has either x or y equal to $+\infty$ or $-\infty$.
- Find the max/min on the "boundary at infinity." At $(0, +\infty)$, f is $+\infty$, so the max on the boundary at infinity is $+\infty$. On the other hand, since $f(x, y) = (x-y)^2 + x^2 - 2x$, if x is either $+\infty$ or $-\infty$, then f is $+\infty$. If x is not $+\infty$ or $-\infty$, then y has to be $+\infty$ or $-\infty$, but then f is again $+\infty$. So the min on the boundary at infinity is also $+\infty$.
- Compare to determine global max/min. Thus there is no global maximum, while the global minimum is -1.

(7) We use the extended 4-step process.

- Find critical points. Note that
$$\nabla f(x,y) = \langle 2x, 2\rangle$$
which is never zero. So there are no critical points.
- Evaluate on critical points. This is unnecessary.
- Find the max/min on the boundary. The boundary is $2x+y^2=3$, so we need to solve a new constrained optimization problem.
  - Find critical points. This means that we need to find Lagrange critical points. The constraint is $g(x,y)=3$, where $g(x,y)=2x+y^2$. Thus the Lagrange critical points happen when $\nabla f(x,y)=\langle 2x,2\rangle$ is parallel to $\nabla g(x,y)=\langle 2,2y\rangle$. Since $\nabla g(x,y)$ is never zero, this happens when there is λ such that $\langle 2x,2\rangle = \lambda\langle 2,2y\rangle$. This means
  $$2x = 2\lambda, \quad 2 = 2\lambda y$$
  Thus, this means $x=\lambda$, so $xy=1$. Using the constraint $2x+y^2=3$, this means $\frac{2}{y}+y^2=3$, or $2+y^3=3y$, or $y^3-3y+2=0$. This can be factorized as $(y-1)(y^2+y-2)=(y-1)^2(y+2)$. Thus this is zero if either $y=1$ or $y=-2$. This means the Lagrange critical poitns are $(1,1)$ and $(-\frac{1}{2},-2)$.
  - Evaluate on critical points. We have $f(1,1)=3$ and $f(-\frac{1}{2},-2)=\frac{1}{4}-4=-\frac{15}{4}$.
  - Find the max/min on the boundary. There is no boundary.
  - Compare to find the max/min candidates. The max candidate is 3, and the min candidate is $-\frac{15}{4}$.
  - Identify the "boundary at infinity." If y is $\pm\infty$, then the constraint says x has to be $-\infty$. If y is finite, then x cannot be infinite, so the boundary at infinity are $(-\infty,+\infty)$ and $(-\infty,-\infty)$.
  - Find the max/min on the "boundary at infinity." At $(-\infty,+\infty)$, f is $+\infty$. At $(-\infty,-\infty)$, note $x=-\frac{y^2}{2}$, so $x^2=\frac{y^4}{4}$, which dominates $2y$, so at $(-\infty,-\infty)$ f is $+\infty$. So the max on the boundary at infinity is $+\infty$, and the min on the boundary at infinity is $+\infty$.
  - Compare to determine global max/min. There is no global maximum, and the global minimum is $-\frac{15}{4}$.
- Compare to find the max/min candidates. There is no max candidate (no global maximum), and the min candidate is $-\frac{15}{4}$.
- Identify the "boundary at infinity." The boundary at infinity would be $x=-\infty$.
- Find the max/min on the "boundary at infinity." At $x=-\infty$, x^2+2y would be always $+\infty$ unless possibly when $y=-\infty$, but again $x\le\frac{3-y^2}{2}$, so x^2 dominates $2y$. So again at $(-\infty,-\infty)$ the value of f is $+\infty$.
- Compare to determine global max/min. So there is no global maximum, and the global minimum is $-\frac{15}{4}$.

(8) We use extended 4-step process.
- Find critical points. This is a constrained optimization, so we need to find Lagrange critical points. Thus we want $\nabla f(x,y)=\langle ye^{xy}, xe^{xy}\rangle$ to be parallel to $\nabla g(x,y)=\langle 3x^2,3y^2\rangle$, where $g(x,y)=16$ is the constraint with $g(x,y)=x^3+y^3$. This happens either when $\nabla g(x,y)$ is zero or there is λ such that $\nabla f(x,y)=\lambda\nabla g(x,y)$.

    - If $\nabla g(x,y) = \langle 0,0\rangle$, then $x = y = 0$, which doesn't satisfy the constraint $x^3 + y^3 = 16$.
    - If $\nabla f(x,y) = \lambda \nabla g(x,y)$, we have

      $$ye^{xy} = 3\lambda x^2, \quad xe^{xy} = 3\lambda y^2,$$

      so

      $$3\lambda x^3 = xye^{xy} = 3\lambda y^3$$

      so either $\lambda = 0$ or $x^3 = y^3$. If $\lambda = 0$, then $ye^{xy} = 0$ and $xe^{xy} = 0$, so $x = y = 0$, which doesn't satisfy the constraint $x^3 + y^3 = 16$. If $x^3 = y^3$, then $x = y$. From the constraint $x^3 + y^3 = 16$, we get $x^3 = 8$, or $x = y = 2$.
  - Evaluate on critical points. At $(2,2)$, we have $f(2,2) = e^4$.
  - Find the max/min on the boundary. There is no boundary.
  - Compare to find the max/min candidates. The max candidate and the min candidate are both e^4.
  - Identify the "boundary at infinity." The boundary at infinity would be $(+\infty, -\infty)$ and $(-\infty, +\infty)$.
  - Find the max/min on the "boundary at infinity." The values on the boundary at infinity is $e^{-\infty} = 0$.
  - Compare to determine global max/min. There is no global min, and the global max is e^4.

(9) We use extended 4-step process.
  - Find critical points. This is a constrained optimization, so we need to find Lagrange critical points. We thus want $\nabla f(x,y,z) = \langle yz, xz, xy\rangle$ to be parallel to $\nabla g(x,y,z) = \langle y+2z, x+2z, 2x+2y\rangle$ where $g(x,y,z) = xy + 2yz + 2zx$ with the constraint being $g(x,y,z) = 12$. This happens either when ∇g is zero or $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$.
    - If ∇g is zero, then $y + 2z = 0$, $x + 2z = 0$, $2x + 2y = 0$. So $x = -y$, so $y + 2z = 0 = -y + 2z$, so $x = y = 0$, and $z = 0$. This is not possible by the constraint $xy + 2yz + 2zx = 12$.
    - If $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$, this means

      $$yz = \lambda(y + 2z), \quad xz = \lambda(x + 2z), \quad xy = \lambda(2x + 2y),$$

      so

      $$xyz = \lambda(xy + 2xz) = \lambda(xy + 2yz) = \lambda(2xz + 2yz)$$

      so $2\lambda xz = 2\lambda yz = \lambda xy$. So either $\lambda = 0$, or $2xz = 2yz = xy$. But $\lambda = 0$ implies $xy = xz = yz = 0$, which is not possible by the constraint $xy+2yz+2zx = 12$. So $2xz = 2yz = xy$, which from constraint implies that $xy = 2yz = 2zx = 4$. So $2z = x = y$, which means $x = y = 2$ and $z = 1$ (because of $x, y, z \geq 0$).
  - Evaluate on critical points. We have $f(2,2,1) = 4$.
  - Find the max/min on the boundary. The boundary is when either x, y, z is zero. If so, the value of f is 0.
  - Compare to find the max/min candidates. The max candidate is 4 and the min candidate is 0.
  - Identify the "boundary at infinity." The boundary at infinity is when either x, y, z is $+\infty$. If $x = +\infty$, then by the constraint we need $y = z = 0$. Similarly for y and z. So the boundary at infinity are $(0, 0, +\infty)$, $(0, +\infty, 0)$ and $(+\infty, 0, 0)$.

- Find the max/min on the "boundary at infinity." At $(0, 0, +\infty)$, $yz \leq 6$, so $xyz = 0$. Similarly for other boundaries at infinity.
- Compare to determine global max/min. The global max is 4, and the global min is 0.

(10) We use extended 4-step process.

- Find critical points. This is a constrained optimization, so we want $\nabla f(x, y, z) = \langle 4, 2, 1\rangle$ to be parallel to $\nabla g(x, y, z) = \langle 2x, 1, 2z\rangle$ where $g(x, y, z) = 1$ is the constraint with $g(x, y, z) = x^2 + y + z^2$. Since ∇g is not zero, there is λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$. This means
$$4 = 2\lambda x, \quad 2 = \lambda, 1 = 2\lambda z$$
So, $\lambda = 2$ says
$$4 = 4x, 1 = 4z,$$
so $x = 1, z = \frac{1}{4}$. From the constraint, this is impossible.
- Evaluate on critical points. This is skipped.
- Find the max/min on the boundary. There is no boundary.
- Compare to find the max/min candidates. There are no max/min candidates, so we know there is no global max/min without going further.

(11) We use extended 4-step process.

- Find critical points. This happens when
$$\nabla f(x, y, z) = \langle \ln(x)+1-\frac{\ln(xyz)}{3}-\frac{x+y+z}{3x}, \ln(y)+1-\frac{\ln(xyz)}{3}-\frac{x+y+z}{3y}, \ln(z)+1-\frac{\ln(xyz)}{3}-\frac{x+y+z}{3z}\rangle$$
is zero. If $x \geq y, z$, we have
$$\ln(x) + 1 - \frac{\ln(xyz)}{3} - \frac{x+y+z}{3x} \geq \ln(x) + 1 - \frac{\ln(x^3)}{3} - \frac{3x}{3x} = 0,$$
so if (x, y, z) is a critical point with $x \geq y, z,$, it should be the case that $x = y = z$. Since given any (x, y, z), either $x \geq y, z, y \geq x, z$ or $z \geq x, y$, we need $x = y = z$ for the critical point to happen. If $x = y = z$, then $\nabla f(x, y, z)$ is zero.
- Evaluate on critical points. Then $f(x, x, x) = 3x \ln(x) - x \ln(x^3) = 0$.
- Find the max/min on the boundary. The boundary is such that $x = 1$ or $y = 1$ or $z = 1$. Since the equation is completely symmetric (namely changing the roles of x, y, z would give the same equation), we only need to consider the case $z = 1$ to compute the max/min on the boundary. The new optimization problem becomes to find the global max/min of
$$f(x, y) = x \ln(x) + y \ln(y) - \frac{x+y+1}{3} \ln(xy)$$
on $x, y \geq 1$. We use extended 4-step process.
  - Find critical points. We want
$$\nabla f(x, y) = \langle \ln(x) + 1 - \frac{\ln(xy)}{3} - \frac{x+y+1}{3x}, \ln(y) + 1 - \frac{\ln(xy)}{3} - \frac{x+y+1}{3y}\rangle$$
to be zero. If $x \geq y$, then
$$\ln(x) + 1 - \frac{\ln(xy)}{3} - \frac{x+y+1}{3x} \geq \ln(x) + 1 - \frac{\ln(x^2)}{3} - \frac{2x+1}{3x} = \frac{\ln(x)}{3} + \frac{1}{3} - \frac{1}{3x} \geq 0,$$

so this is zero if $x = y = 1$. The same conclusion applies when $y \geq x$. Thus, the critical point would be $(1, 1)$.

  - Evaluate on critical points. On the critical point, $f(1, 1) = 0$.
  - Find the max/min on the boundary. The boundary would be $x = 1$ or $y = 1$. Since the equation is completely symmetric, we only need to consider the case $y = 1$ to compute the max/min on the boundary. The new optimization problem becomes to find the global max/min of

$$f(x) = x\ln(x) - \frac{x+2}{3}\ln(x) = \frac{2x-2}{3}\ln(x)$$

    on $x \geq 1$. This is a standard optimization problem in one-variable calculus: we evaluate at the boundary $x = 1$ which is $f(1) = 0$, find the critical point $f'(x) = 0$ which is

$$\frac{2}{3}\ln(x) + \frac{2x-2}{3x} = 0,$$

    which can only happen when $x = 1$ (otherwise each term on the left is strictly positive), at which $f(1) = 0$, and see what happens as $x \to +\infty$, at which $f(x) \to +\infty$. So the max does not exist, and the min is 0.
  - Compare to find the max/min candidates. The max candidate and the min candidate are both 0.
  - Identify the "boundary at infinity." The boundary at infinity would be either x or y being $+\infty$. Then f goes to $+\infty$, as either $x\ln(x)$ or $y\ln(y)$ is the major term.
  - Find the max/min on the "boundary at infinity." So the max/min on the boundary at infinity is $+\infty$.
  - Compare to determine global max/min. There is no global max, and the global min is 0.
- Compare to find the max/min candidates. The max and min candidates are both 0.
- Identify the "boundary at infinity." The boundary at infinity is either x, y, z is $+\infty$.
- Find the max/min on the "boundary at infinity." Then f goes to $+\infty$, as either $x\ln(x)$, $y\ln(y)$ or $z\ln(z)$ is the major term.
- Compare to determine global max/min. The global max doesn't exist, and the global min is 0.

□

Exercise 3. Determine whether there is a global maximum or a global minimum of a function f on a region D, and if they exist, find the values.

(1) $f(x, y, z) = z$ on $D = \{(x, y, z) \mid x^2 + y^2 = z^2,\ x + y + z = 24\}$
(2) $f(x, y, z) = x + y + z$ on $D = \{(x, y, z) \mid x^2 + z^2 \leq 2,\ x + y \leq 1\}$
(3) $f(x, y, z) = x^2 + y^2 + z^2$ on $D = \{(x, y, z) \mid x - y = 1,\ y^2 - z^2 = 1\}$
(4) $f(x, y, z) = yz + xy$ on $D = \{(x, y, z) \mid xy = 1,\ y^2 + z^2 \leq 1\}$
(5) $f(x, y, z) = x^2 + 2y^2 + 3z^2$ on $D = \{(x, y, z) \mid x + y + z = 1,\ x - y + 2z = 2\}$
(6) $f(x, y, z) = x^2 + y^2 + z^2$ on $D = \{(x, y, z) \mid 2x + y + 2z = 9,\ 5x + 5y + 7z = 29\}$
(7) $f(x, y, z) = x^2 + y^2 + z^2$ on $D = \{(x, y, z) \mid z^2 = x^2 + y^2,\ x + y - z + 1 = 0\}$
(8) $f(x, y, z) = xyz$ on $D = \{(x, y, z) \mid x + y + z = 1,\ x + y - z = 0\}$

Solution.

(1) Since whether the region is compact or not is not clear, we use the extended 4-step process.

- Step 1. Find the Lagrange critical points. The constraints are $g(x,y,z) = 0$ and $h(x,y,z) = 24$ where $g(x,y,z) = x^2+y^2-z^2$ and $h(x,y,z) = x+y+z$. Lagrange critical points are when either $\nabla g(x,y,z) = \langle 2x, 2y, -2z\rangle$ or $\nabla h(x,y,z) = \langle 1,1,1\rangle$ is zero, or $\nabla f(x,y,z) = \langle 0,0,1\rangle$ is $\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z)$. Note that $\nabla h(x,y,z)$ is never zero, and $\nabla g(x,y,z)$ is zero if $x = y = z = 0$, which does not satisfy $x+y+z = 24$. Thus we need to solve

$$\nabla f(x,y,z) = \lambda\nabla g(x,y,z) + \mu\nabla h(x,y,z)$$

or

$$0 = 2\lambda x + \mu, \quad 0 = 2\lambda y + \mu, \quad 1 = -2\lambda z + \mu$$

From the first two equations, $2\lambda x = 2\lambda y$, so either $\lambda = 0$ or $x = y$.
  - If $\lambda = 0$, we have $\mu = 0$ and $\mu = 1$, which is a contradiction.
  - If $x = y$, then the constraints say $z^2 = 2x^2$ and $2x+z = 24$, so

$$2x^2 = z^2 = (24-2x)^2 = 4x^2 - 96x + 576$$

  or $2x^2 - 96x + 576 = 0$, or $x^2 - 48x + 288 = 0$. Thus $(x-24)^2 = 288$, or $x - 24 = 12\sqrt{2}$ or $-12\sqrt{2}$, or $x = 24+12\sqrt{2}$ or $x = 24-12\sqrt{2}$. Thus the Lagrange critical points are $(24+12\sqrt{2}, 24+12\sqrt{2}, -24-24\sqrt{2})$ and $(24-12\sqrt{2}, 24-12\sqrt{2}, -24+24\sqrt{2})$.
- Step 2. Evaluate on the Lagrange critical points. We have $f(24+12\sqrt{2}, 24+12\sqrt{2}, -24-24\sqrt{2}) = -24-24\sqrt{2}$ and $f(24-12\sqrt{2}, 24-12\sqrt{2}, -24+24\sqrt{2}) = -24+24\sqrt{2}$.
- Step 3. Evaluate on the boundary. There is no boundary.
- Step 4. Find the max/min candidates. The max candidate is $-24+24\sqrt{2}$ and the min candidate is $-24-24\sqrt{2}$.
- Step 5. Find the boundary at infinity. Note that $x^2+y^2 = z^2 = (24-x-y)^2$, so

$$x^2+y^2 = x^2+y^2+576-48x-48y+2xy$$

or

$$2xy - 48x - 48y + 576 = 0$$

or

$$xy - 24x - 24y + 288 = 0$$

or

$$(x-24)(y-24) = 288$$

so definitely x or y can be infinite. If $x = \pm\infty$, then $y - 24$ has to be 0, and if $y = \pm\infty$, then $x - 24$ has to be 0. So the boundary at infinity are $(+\infty, 24, -\infty)$, $(-\infty, 24, +\infty)$, $(24, +\infty, -\infty)$, $(24, -\infty, +\infty)$.
- Step 6. Find the max/min on the boundary at infinity. This would be $+\infty$ or $-\infty$.
- Step 7. Compare. There are no global max and no global min.

(2) Since the region is not compact, we use the extended 4-step process.

- Step 1. Find the critical points. Since $\nabla f(x,y,z) = \langle 1,1,1\rangle$, this is never zero.
- Step 2. Evaluate on the critical points. This is skipped.
- Step 3. Evaluate on the boundary. The boundary is consisted of two parts, $\{(x,y,z) \mid x^2+z^2 = 2,\ x+y \leq 1\}$ and $\{(x,y,z) \mid x^2+z^2 \leq 2,\ x+y = 1\}$.

- On the boundary $\{(x, y, z) \mid x^2 + z^2 = 2,\ x + y \leq 1\}$, we use the extended 4-step process.
  * Step 1. Find the Lagrange critical points. The constraint is $g(x, y, z) = 2$ where $g(x, y, z) = x^2 + z^2$. Thus we want $\nabla f(x, y, z) = \langle 1, 1, 1\rangle$ to be parallel to $\nabla g(x, y, z) = \langle 2x, 0, 2z\rangle$. This is only possible if $x = z = 0$, which does not satisfy the constraint $x^2 + z^2 = 2$.
  * Step 2. Evaluate on the Lagrange critical points. This is skipped.
  * Step 3. Evaluate on the boundary. The boundary is the region $\{(x, y, z) \mid x^2 + z^2 = 2,\ x + y = 1\}$. Since this is compact, we use the 4-step process.
    · Step 1. Find the Lagrange critical points. The constraints are $g(x, y, z) = 2$ and $h(x, y, z) = 1$ where $h(x, y, z) = x + y$. This means $\langle 1, 1, 1\rangle$, $\langle 2x, 0, 2z\rangle$ and $\langle 1, 1, 0\rangle$ form a plane. This is only possible if $2x = 0$, or $x = 0$. From the constraints, we have $y = 1$ and $z^2 = 2$, so the Lagrange critical points are $(0, 1, \sqrt{2})$ and $(0, 1, -\sqrt{2})$.
    · Step 2. Evaluate on the Lagrange critical points. This is $f(0, 1, \sqrt{2}) = 1 + \sqrt{2}$ and $f(0, 1, -\sqrt{2}) = 1 - \sqrt{2}$.
    · Step 3. Evaluate on the boundary. There is no boundary.
    · Step 4. Compare. The max is $1 + \sqrt{2}$, and the min is $1 - \sqrt{2}$.
  * Step 4. Find the max/min candidate. The max candidate is $1 + \sqrt{2}$, and the min candidate is $1 - \sqrt{2}$.
  * Step 5. Find the boundary at infinity. This is only possible if $y = -\infty$.
  * Step 6. Find the max/min on the boundary at infinity. Since x, z are finite, $f(x, -\infty, z) = -\infty$.
  * Step 7. Compare. The global max is $1 + \sqrt{2}$, and there is no global min.
- On the boundary $\{(x, y, z) \mid x^2 + z^2 \leq 2,\ x + y = 1\}$, this region is compact, so we use the 4-step process.
  * Step 1. Find the Lagrange critical points. This is possible if $\nabla f(x, y, z)$ is parallel to $\nabla h(x, y, z)$, but $\langle 1, 1, 1\rangle$ and $\langle 1, 1, 0\rangle$ is never parallel.
  * Step 2. Evaluate on the Lagrange critical points. This is skipped.
  * Step 3. Find the max/min on the boundary. This is already done above. The max is $1 + \sqrt{2}$, and the min is $1 - \sqrt{2}$.
  * Step 4. Compare. The global max is $1 + \sqrt{2}$, and the global min is $1 - \sqrt{2}$.

• Step 4. Find the max/min candidates. The max candidate is $1 + \sqrt{2}$, and the min candidate is $1 - \sqrt{2}$.
• Step 5. Find the boundary at infinity. This is only possible if $y = -\infty$.
• Step 6. Find the max/min on the boundary at infinity. This is $-\infty$.
• Step 7. Compare. The global max is $1 + \sqrt{2}$, and there is no global min.

(3) Since the region is not compact, we use the extended 4-step process.
• Step 1. Find the Lagrange critical points. The constraints are $g(x, y, z) = 1$ and $h(x, y, z) = 1$ where $g(x, y, z) = x - y$ and $h(x, y, z) = y^2 - z^2$. Thus we want $\nabla f(x, y, z) = \langle 2x, 2y, 2z\rangle$, $\nabla g(x, y, z) = \langle 1, -1, 0\rangle$ and $\nabla h(x, y, z) = \langle 0, 2y, -2z\rangle$ are on a plane. This is the case either when $\nabla g(x, y, z)$ or $\nabla h(x, y, z)$ is zero, or when $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$. If the former case happens, then

$y = z = 0$, which does not satisfy $y^2 - z^2 = 1$. Thus we want

$$2x = \lambda, \quad 2y = -\lambda + 2\mu y, \quad 2z = -2\mu z$$

or

$$2y = -2x + 2\mu y, \quad 2z = -2\mu z$$

From the third equation, either $\mu = -1$ or $z = 0$.

- If $\mu = -1$, we have $2y = -2x - 2y$, so $4y = -2x$, or $x = -2y$. From $x - y = 1$, $-3y = 1$, or $y = -\frac{1}{3}$. So $\frac{1}{9} - z^2 = 1$, which is a contradiction.
- If $z = 0$, we have $y^2 = 1$, so $y = 1$ or -1. Then $x = 2$ or 0. Thus the Lagrange critical points are $(2, 1, 0)$ and $(0, -1, 0)$.

- Step 2. Evaluate on the Lagrange critical points. We have $f(2, 1, 0) = 5$ and $f(0, -1, 0) = 1$.
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Find the max/min candidate. The max candidate is 5, and the min candidate is 0.
- Step 5. Find the boundary at infinity. If any of x, y, z is finite, all of them are finite. The only requirement is x, y have the same sign, so the boundary at infinity are $(+\infty, +\infty, +\infty)$, $(+\infty, +\infty, -\infty)$, $(-\infty, -\infty, +\infty)$, $(-\infty, -\infty, -\infty)$.
- Step 6. Find the max/min on the boundary at infinity. This is $+\infty$.
- Step 7. Compare. There is no global max, and the global min is 1.

(4) Since the region is not compact, we need to use the extended 4-step process.

- Step 1. Find the Lagrange critical points. The constraint is $g(x, y, z) = 1$, where $g(x, y, z) = xy$. Thus we want $\nabla f(x, y, z) = \langle y, x + z, y\rangle$ is parallel to $\nabla g(x, y, z) = \langle y, x, 0\rangle$. Thus $y = 0$, which is impossible due to $xy = 1$.
- Step 2. Evaluate on the Lagrange critical points. This is skipped.
- Step 3. Find the max/min on the boundary. The boundary is the region

$$\{(x, y, z) \mid xy = 1,\ y^2 + z^2 = 1\}$$

This is again not compact, so we use the extended universal straetgy.

- Step 1. Find the Lagrange critical points. The constraints are $g(x, y, z) = 1$ and $h(x, y, z) = 1$ where $h(x, y, z) = y^2 + z^2$. Thus we want either $\nabla g(x, y, z) = \langle y, x, 0\rangle$ or $\nabla h(x, y, z) = \langle 0, 2y, 2z\rangle$ is zero, or $\nabla f(x, y, z) = \lambda\nabla g(x, y, z) + \mu\nabla h(x, y, z)$. The former case happens either $x = y = 0$ or $y = z = 0$, which do not satisfy $xy = 1$. Thus we need to solve

$$y = \lambda y, \quad x + z = \lambda x + 2\mu y, \quad y = 2\mu z$$

From the first equation, either $\lambda = 1$ or $y = 0$.

* If $\lambda = 1$, we have

$$x + z = x + 2\mu y, \quad y = 2\mu z$$

or

$$z = 2\mu y, \quad y = 2\mu z$$

or $z = 2\mu y = 4\mu^2 z$, so either $z = 0$ or $4\mu^2 = 1$.

· If $z = 0$, then $y^2 = 1$, so $y = 1$ or -1, so $x = 1$ or -1. Thus the Lagrange critical points are $(1, 1, 0)$ or $(-1, -1, 0)$.

  - If $4\mu^2 = 1$, then $\mu = \frac{1}{2}$ or $\mu = -\frac{1}{2}$. If $\mu = \frac{1}{2}$, $z = y$, while if $\mu = -\frac{1}{2}$, $z = -y$. In any case, $y^2 + z^2 = 1$ means $2y^2 = 1$, so $y = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$. Thus using $xy = 1$, the Lagrange critical points are $(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

  * If $y = 0$, $xy = 1$ is not satisfied.

- Step 2. Evaluate on the Lagrange critical points. We have $f(1, 1, 0) = 1$, $f(-1, -1, 0) = 1$, $f(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{3}{2}$, $f(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2}$, $f(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2}$, $f(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{3}{2}$.
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Compare to get the max/min candidate. The max candidate is $\frac{3}{2}$ and the min candidate is $\frac{1}{2}$.
- Step 5. Find the boundary at infinity. The boundary at infinity can only happen when $x = \pm\infty$, where $y = 0$ and $z^2 = 1$. Thus the boundary at infinity is $(\pm\infty, 0, \pm 1)$.
- Step 6. Find the max/min on the boundary at infinity. This is 1, because $xy = 1$.
- Step 7. Compare. The global max is $\frac{3}{2}$ and the global min is $\frac{1}{2}$.

- Step 4. Compare to get the max/min candidate. The max candidate is $\frac{3}{2}$ and the min candidate is $\frac{1}{2}$.
- Step 5. Find the boundary at infinity. The boundary at infinity can happen when $x = \pm\infty$, for which $y = 0$ and $z^2 \leq 1$.
- Step 6. Find the max/min on the boundary at infinity. This is 1, because $xy = 1$.
- Step 7. Compare. The global max is $\frac{3}{2}$, and the global min is $\frac{1}{2}$.

(5) Since the region is not compact, we use the extended 4-step process.

- Step 1. Find the Lagrange critical points. The constraints are $g(x, y, z) = 1$ and $h(x, y, z) = 2$, where $g(x, y, z) = x + y + z$ and $h(x, y, z) = x - y + 2z$. We want either $\nabla g(x, y, z) = \langle 1, 1, 1\rangle$ or $\nabla h(x, y, z) = \langle 1, -1, 2\rangle$ is zero, or $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$. Since the former is impossible, we want

$$2x = \lambda + \mu, \quad 4y = \lambda - \mu, \quad 6z = \lambda + 2\mu$$

or

$$x = \frac{\lambda + \mu}{2}, \quad y = \frac{\lambda - \mu}{4}, \quad z = \frac{\lambda + 2\mu}{6}$$

Plugging these into constraints, we get

$$1 = x + y + z = \frac{\lambda + \mu}{2} + \frac{\lambda - \mu}{4} + \frac{\lambda + 2\mu}{6} = \frac{11}{12}\lambda + \frac{7}{12}\mu$$

$$2 = x - y + 2z = \frac{\lambda + \mu}{2} - \frac{\lambda - \mu}{4} + \frac{\lambda + 2\mu}{3} = \frac{7}{12}\lambda + \frac{17}{12}\mu$$

Thus

$$7 = \frac{77}{12}\lambda + \frac{49}{12}\mu$$

$$22 = \frac{77}{12}\lambda + \frac{187}{12}\mu$$

so

$$15 = \frac{138}{12}\mu = \frac{23}{2}\mu$$

or $\mu = \frac{30}{23}$. Thus
$$1 = \frac{11}{12}\lambda + \frac{35}{46}$$
or $\frac{11}{12}\lambda = \frac{11}{46}$ or $\lambda = \frac{6}{23}$. Thus
$$x = \frac{18}{23}, \quad y = -\frac{6}{23}, \quad z = \frac{11}{23}$$

- Step 2. Evaluate on the Lagrange critical points. We have
$$f(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}) = \frac{324 + 72 + 363}{529} = \frac{759}{529} = \frac{33}{23}$$
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Find the max/min candidate. The max candidate and the min candidate are both $\frac{33}{23}$.
- Step 5. Find the boundary at infinity. Since $x + y + z = 1$ and $x - y + 2z = 2$ gives $2x + 3z = 3$, x and z have opposite signs. Also, we have $2y - z = -1$, so y, z have the same signs. So the boundary at infinity are $(+\infty, -\infty, -\infty)$ and $(-\infty, +\infty, +\infty)$.
- Step 6. Evaluate on the boundary at infinity. This is $+\infty$.
- Step 7. Compare. There is no global max, and the global min is $\frac{33}{23}$.

(6) Since the region is not compact, we use the extended 4-step process.

- Step 1. Find the Lagrange critical points. The constraints are $g(x, y, z) = 9$ and $h(x, y, z) = 29$, where $g(x, y, z) = 2x + y + 2z$ and $h(x, y, z) = 5x + 5y + 7z$. Thus $\nabla g(x, y, z) = \langle 2, 1, 2\rangle$ and $h(x, y, z) = \langle 5, 5, 7\rangle$. They are never zero, so for the Lagrange critical points we need
$$\langle 2x, 2y, 2z\rangle = \lambda\langle 2, 1, 2\rangle + \mu\langle 5, 5, 7\rangle$$
or
$$2x = 2\lambda + 5\mu, \quad 2y = \lambda + 5\mu, \quad 2z = 2\lambda + 7\mu$$
Plugging into the constraints, we get
$$(2\lambda + 5\mu) + \frac{\lambda + 5\mu}{2} + (2\lambda + 7\mu) = 9$$
$$\frac{5}{2}(2\lambda + 5\mu) + \frac{5}{2}(\lambda + 5\mu) + \frac{7}{2}(2\lambda + 7\mu) = 29$$
or
$$\frac{9}{2}\lambda + \frac{29}{2}\mu = 9$$
$$\frac{29}{2}\lambda + \frac{99}{2}\mu = 29$$
so
$$\frac{1}{2}\lambda + \frac{29}{18}\mu = 1$$
$$\frac{1}{2}\lambda + \frac{99}{58}\mu = 1$$
so $\mu = 0$ and $\lambda = 2$. This gives $x = 2$, $y = 1$, $z = 2$.
- Step 2. Evaluate on the critical points. We have $f(2, 1, 2) = 9$.
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Find the max/min candidate. The max candidate is 9 and the min candidate is 9.

- Step 5. Find the boundary at infinity. We have $10x+5y+10z=45$, so $5x+3z=16$, so x,z have the opposite signs. Also, $5x+\frac{5}{2}y+5z=\frac{45}{2}$, so $\frac{5}{2}y+2z=\frac{13}{2}$, so y,z have the opposite signs. So the boundary at infinity is $(+\infty,+\infty,-\infty)$ and $(-\infty,-\infty,+\infty)$.
- Step 6. Evaluate on the boundary at infinity. This is $+\infty$.
- Step 7. Compare. There is no global max, and the global min is 9.

(7) Since the region is not compact, we use the extended 4-step process.

- Step 1. Find the Lagrange critical points. The constraints are $g(x,y,z)=0$ and $h(x,y,z)=0$ where $g(x,y,z)=z^2-x^2-y^2$ and $h(x,y,z)=x+y-z+1$. Thus

$$\nabla g(x,y,z)=\langle -2x,-2y,2z\rangle,\quad \nabla h(x,y,z)=\langle 1,1,-1\rangle$$

One of these can be zero if $x=y=z$, which does not satisfy the constraints $x+y-z+1=0$. Thus for the Lagrange critical points we need

$$2x=-2\lambda x+\mu,\quad 2y=-2\lambda y+\mu,\quad 2z=2\lambda z-\mu$$

Subtracting the second equation from the first equation, we get

$$2(x-y)=-2\lambda(x-y),$$

so either $x-y=0$ or $\lambda=-1$.

  - If $x-y=0$, then from the constraints we get

$$z^2=2x^2,\quad 2x-z+1=0,$$

so $z=2x+1$, so $2x^2=z^2=(2x+1)^2=4x^2+4x+1$, or $2x^2+4x+1=0$. So $x=\frac{-4\pm\sqrt{16-8}}{4}=\frac{-2\pm\sqrt{2}}{2}$. Thus the Lagrange critical points are $(\frac{-2+\sqrt{2}}{2},\frac{-2+\sqrt{2}}{2},-1+\sqrt{2})$ and $(\frac{-2-\sqrt{2}}{2},\frac{-2-\sqrt{2}}{2},-1-\sqrt{2})$.
  - If $\lambda=-1$, we have

$$2x=2x+\mu,\quad 2y=2y+\mu,\quad 2z=-2z-\mu,$$

so $\mu=0$ and $z=0$. Thus $x=y=0$, which does not satisfy $x+y-z+1=0$.

- Step 2. Evaluate on the critical points. We have $f(\frac{-2+\sqrt{2}}{2},\frac{-2+\sqrt{2}}{2},-1+\sqrt{2})=6-4\sqrt{2}$ and $f(\frac{-2-\sqrt{2}}{2},\frac{-2-\sqrt{2}}{2},-1-\sqrt{2})=6+4\sqrt{2}$.
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Find the max/min candidate. The max candidate is $6+4\sqrt{2}$, and the min candidate is $6-4\sqrt{2}$.
- Step 5. Find the boundary at infinity. On the boundary at infinity, z must be infinite.
- Step 6. Evaluate on the boundary at infinity. On the boundary at infinity, f is $+\infty$.
- Step 7. Compare. There is no global max, and the global min is $6-4\sqrt{2}$.

(8) Since the region is not compact, we use the extended 4-step process.

- Step 1. Find the Lagrange critical points. The constraints are $g(x,y,z)=1$ and $h(x,y,z)=0$, where $g(x,y,z)=x+y+z$ and $h(x,y,z)=x+y-z$. Thus $\nabla g(x,y,z)$ and $\nabla h(x,y,z)$ are never zero, so we need

$$\nabla f(x,y,z)=\lambda\nabla g(x,y,z)+\mu\nabla h(x,y,z),$$

or

$$yz=\lambda+\mu,\quad xz=\lambda+\mu,\quad xy=\lambda-\mu$$

From the first and the second equations, we have $yz=xz$, so either $y=x$ or $z=0$.

    - If $y = x$, from the constraints, $2x + z = 1$ and $2x - z = 0$. Adding them we get $4x = 1$, or $x = \frac{1}{4}$, and $z = 2x = \frac{1}{2}$. Thus $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is a Lagrange critical point.
    - If $z = 0$, from the constraints $x + y = 1$ and $x + y = 0$, which is contradictory.
- Step 2. Evaluate on the critical points. We have $f(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) = \frac{1}{32}$.
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Find the max/min candidate. The max candidate is $\frac{1}{32}$ and the min candidate is $\frac{1}{32}$.
- Step 5. Find the boundary at infinity. Adding the two constraints, we get $2x + 2y = 1$, so x and y have opposite signs. Subtracting the second constraint from the first constraint, we get $2z = 1$. So $z = \frac{1}{2}$. So the boundary at infinity are $(+\infty, -\infty, \frac{1}{2})$ and $(-\infty, +\infty, \frac{1}{2})$.
- Step 6. Evaluate on the boundary at infinity. On both points, the value of f is $-\infty$.
- Step 7. Compare. The global max is $\frac{1}{32}$, and there is no global min.

□

Exercise 4. Find the distance between two objects.

(1) The distance between the surface $xy^2z^3 = 2$ and the origin.
(2) The distance between the surface $z = x^2 + y^2$ and the point $(1, 1, 0)$.

Solution.

(1) The constrained optimization problem is to find the global minimum of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ given the constraint $g(x, y, z) = 2$, where $g(x, y, z) = xy^2z^3$. Since the region is not compact, we use the extended 4-step process.
- Step 1. Find the Lagrange critical points. We want $\nabla f(x, y, z) = \langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \rangle$ to be parallel to $\nabla g(x, y, z) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$. This is possible if either $\nabla g(x, y, z)$ is zero or $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$.
    - If $\nabla g(x, y, z) = \langle 0, 0, 0 \rangle$, $y^2z^3 = 0$, so either $y = 0$ or $z = 0$. This is impossible because of the constraint $xy^2z^3 = 2$.
    - If $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$, then

$$\frac{x}{\sqrt{x^2 + y^2 + z^2}} = \lambda y^2 z^3, \quad \frac{y}{\sqrt{x^2 + y^2 + z^2}} = 2\lambda xyz^3, \quad \frac{z}{\sqrt{x^2 + y^2 + z^2}} = 3\lambda xy^2 z^2$$

Thus

$$\lambda xy^2z^3 = \frac{x^2}{\sqrt{x^2 + y^2 + z^2}} = \frac{y^2}{2\sqrt{x^2 + y^2 + z^2}} = \frac{z^2}{3\sqrt{x^2 + y^2 + z^2}},$$

or $x^2 = \frac{y^2}{2} = \frac{z^2}{3}$. Thus $y^2 = 2x^2$, and $z = \pm\sqrt{3}x$. Putting them into the constraint, we get $\pm 6\sqrt{3}x^6 = 2$. Thus, $z = \sqrt{3}x$, and $6\sqrt{3}x^6 = 2$, sor $x^6 = \frac{1}{3\sqrt{3}}$, or $x = \pm\frac{1}{3^{1/4}}$. Thus the Lagrange critical points are $(\frac{1}{3^{1/4}}, \pm\frac{\sqrt{2}}{3^{1/4}}, 3^{1/4})$ and $(-\frac{1}{3^{1/4}}, \pm\frac{\sqrt{2}}{3^{1/4}}, -3^{1/4})$.
- Step 2. Evaluate on the Lagrange critical points. We have $f(\pm\frac{1}{3^{1/4}}, \pm\frac{\sqrt{2}}{3^{1/4}}, \pm 3^{1/4}) = \sqrt{\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \sqrt{3}} = \sqrt{2\sqrt{3}}$.
- Step 3. Find the max/min on the boundary. There is no boundary.

- Step 4. Find the max/min candidates. The max candidate and the min candidate are both $\sqrt{2\sqrt{3}}$.
- Step 5. Find the boundary at infinity. Obviously either x, y, z is $\pm\infty$.
- Step 6. Find the max/min on the boundary at infinity. Then $f = +\infty$.
- Step 7. Compare. The global min is $\sqrt{2\sqrt{3}}$.

(2) The constrained optimization problem is to find the global minimum of $f(x, y, z) = \sqrt{(x-1)^2 + (y-1)^2 + z^2}$ given the constraint $g(x, y, z) = 0$ where $g(x, y, z) = z - x^2 - y^2$. Since the region is not compact, we use the extended 4-step process.

- Step 1. Find the Lagrange critical points. We want

$$\nabla f(x,y,z) = \langle \frac{x-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}}, \frac{y-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}}, \frac{z}{\sqrt{(x-1)^2+(y-1)^2+z^2}} \rangle$$

to be parallel to $\nabla g(x, y, z) = \langle -2x, -2y, 1\rangle$. Since $\nabla g(x, y, z)$ is never zero, we need to solve $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$. This is

$$\frac{x-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = -2\lambda x, \quad \frac{y-1}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = -2\lambda y,$$
$$\frac{z}{\sqrt{(x-1)^2+(y-1)^2+z^2}} = \lambda$$

From the first equation, $x = 0$ does not work, as the left side is not 0 if $x = 0$. Similarly, $y = 0$ does not work. Thus we can divide by x or y, so we have

$$-\frac{x-1}{2x} = -\frac{y-1}{2y} = z$$

or

$$z = -\frac{1}{2} + \frac{1}{2x} = -\frac{1}{2} + \frac{1}{2y}$$

Thus $x = y$, and $z = -\frac{1}{2} + \frac{1}{2x}$. Thus the constraint becomes $z = 2x^2$, so

$$2x^2 = z = -\frac{1}{2} + \frac{1}{2x}$$

or

$$4x^3 = -x + 1$$

or $4x^3 + x - 1 = 0$. Since $x = \frac{1}{2}$ is a root, this can be factorized as $(2x-1)(2x^2+x+1) = 0$. Thus, either $x = \frac{1}{2}$ or $2x^2 + x + 1 = 0$. Since the latter quadratic has discriminant $1 - 8 = -7 < 0$, this has no root. Thus $x = \frac{1}{2} = y$ and $z = \frac{1}{2}$. Thus $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is a Lagrange critical point.
- Step 2. Evaluate on the Lagrange critical points. We have $f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{\sqrt{3}}{2}$.
- Step 3. Find the max/min on the boundary. There is no boundary.
- Step 4. Find the max/min candidates. The max candidate and the min candidate are both $\frac{\sqrt{3}}{2}$.
- Step 5. Find the boundary at infinity. Obviously either x, y, z is $\pm\infty$.
- Step 6. Find the max/min on the boundary at infinity. Then the value of f is $+\infty$.
- Step 7. Compare. The global minimum is $\frac{\sqrt{3}}{2}$.

$\square$